

MATH 209

Fundamental Mathematics II

Winter 2015

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Office Hours: Thursday from 12:00-14:00 in LB 541-4



3. Limits and the derivative

- 1 3-1 Introduction to limits
- 2 3-2 Infinite Limits and Limits at Infinity
- 3 3-3 Continuity
- 4 3-4 The Derivative
- 5 3-5 Basic Differentiation Properties
- 6 3-6 Differentials
- 7 3-7 Marginal Analysis in Business and Economics



3. Limits and the derivative

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3 3-3 Continuity

4 3-4 The Derivative

5 3-5 Basic Differentiation Properties

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7 3-7 Marginal Analysis in Business and Economics



3. Limits and the derivative

3-1 Introduction to limits

Learning Objectives

- Evaluate limits of function from graphs
- Evaluate limits of functions algebraically



3. Limits and the derivative

3-1 Introduction to limits

DEFINITION **limit**

- Symbol:

$$\lim_{x \rightarrow c} f(x) = L$$

- Spoken: "The limit as x approaches c of $f(x)$ is L "

- Symbol:

$$f(x) \rightarrow L \quad \text{as} \quad x \rightarrow c$$

- Spoken: " $f(x)$ approaches L as x approaches c "

- Usage:
 - x is a variable
 - f is a function
 - c is a real number
 - L is a real number

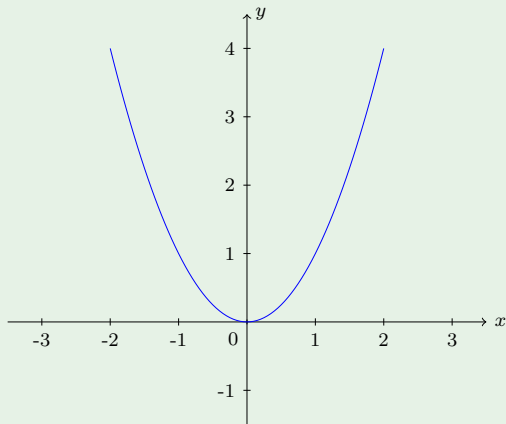
- Meaning: As x gets closer to c , but not equal to c , the values of $f(x)$ get closer and closer to L .



3. Limits and the derivative

3-1 Introduction to limits

Example:

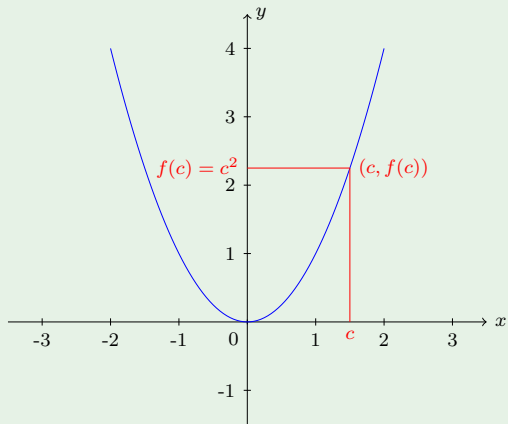


$$f(x) = x^2$$

3. Limits and the derivative

3-1 Introduction to limits

Example:

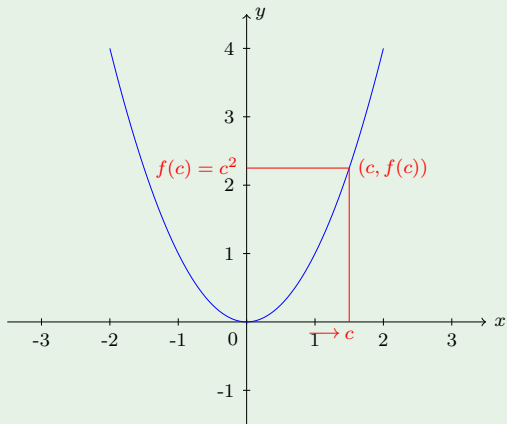


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3. Limits and the derivative

3-1 Introduction to limits

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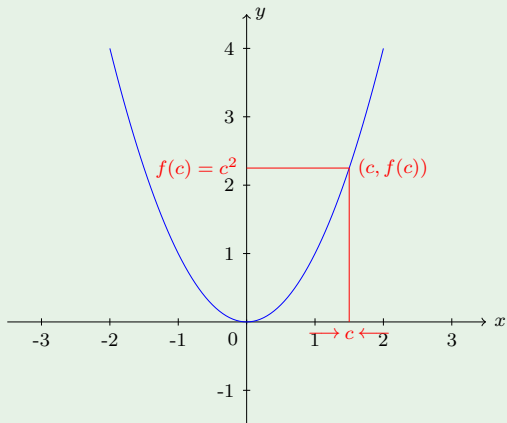


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3. Limits and the derivative

3-1 Introduction to limits

Example:

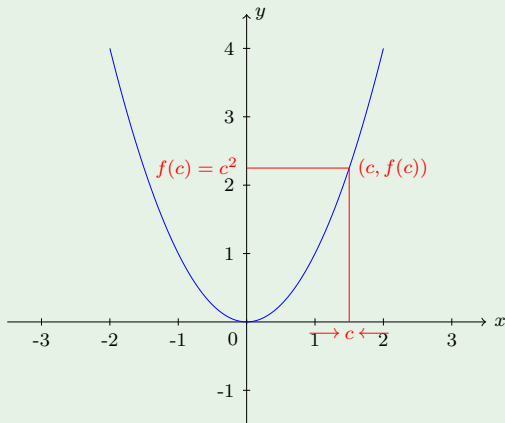


$$f(x) = x^2$$

3. Limits and the derivative

3-1 Introduction to limits

Example:



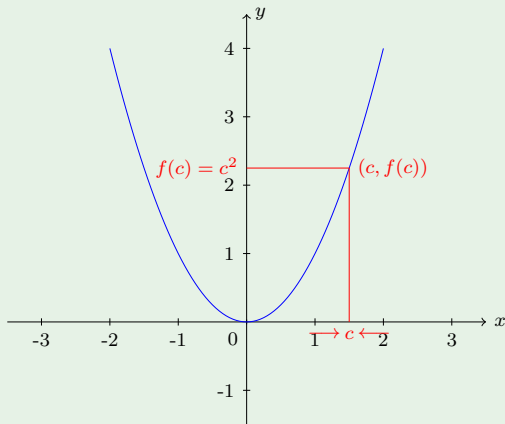
$$f(x) = x^2$$

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} x^2 = c^2$$

3. Limits and the derivative

3-1 Introduction to limits

Example:



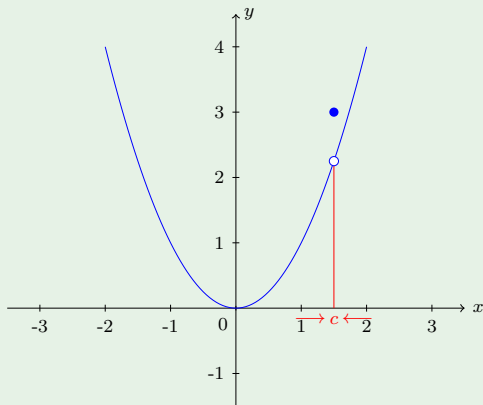
$$f(x) = x^2$$

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} x^2 = c^2 = f(c)$$

3. Limits and the derivative

3-1 Introduction to limits

Example:

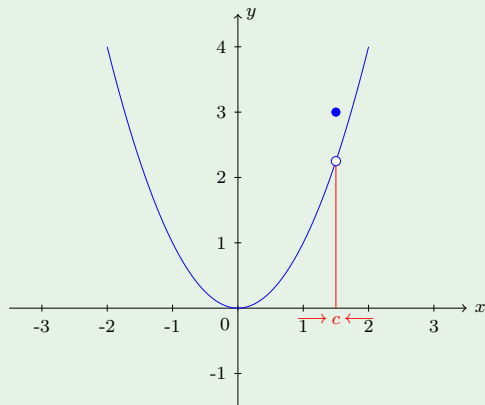


$$f(x) = \begin{cases} x^2 & \text{if } x \neq c \\ 3 & \text{if } x = c \end{cases}$$

3. Limits and the derivative

3-1 Introduction to limits

Example:



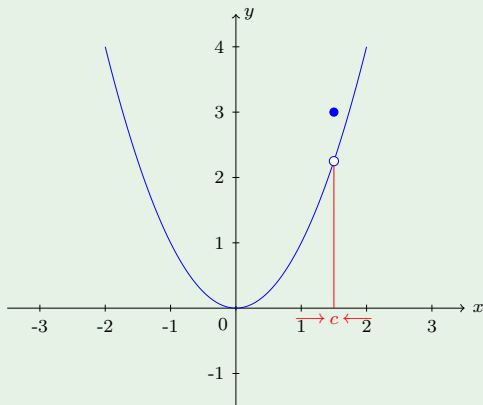
$$f(x) = \begin{cases} x^2 & \text{if } x \neq c \\ 3 & \text{if } x = c \end{cases}$$

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c^-} f(x) = \lim_{x \rightarrow c^+} f(x) = c^2$$

3. Limits and the derivative

3-1 Introduction to limits

Example:



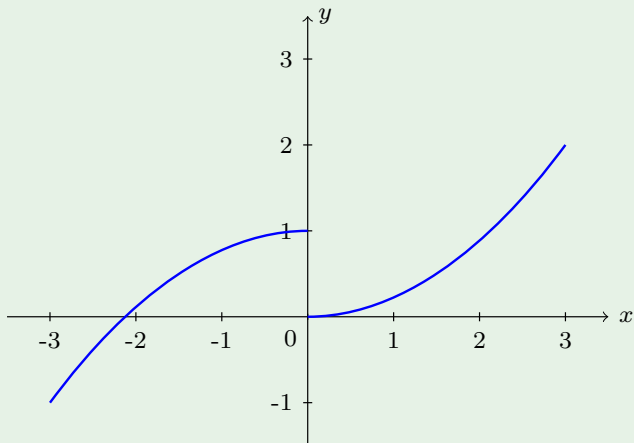
$$f(x) = \begin{cases} x^2 & \text{if } x \neq c \\ 3 & \text{if } x = c \end{cases}$$

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c^-} f(x) = \lim_{x \rightarrow c^+} f(x) = c^2 \neq f(c) = 3$$

3. Limits and the derivative

3-1 Introduction to limits

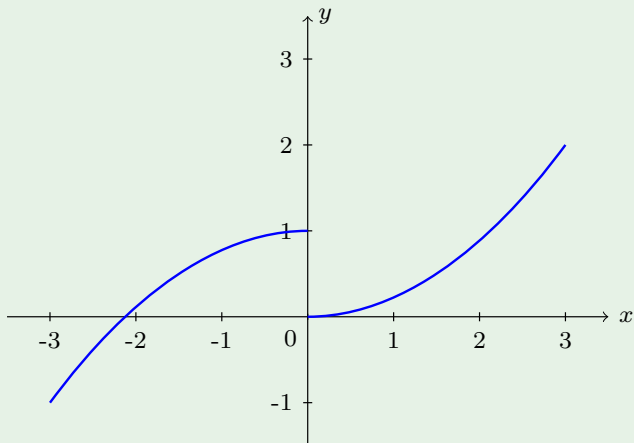
Example: Situation where the limit does not exist



3. Limits and the derivative

3-1 Introduction to limits

Example: Situation where the limit does not exist

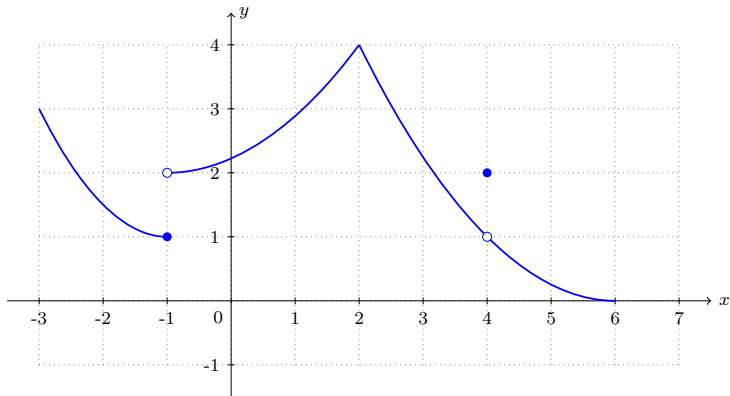


$$\lim_{x \rightarrow 0} f(x) = DNE$$

3. Limits and the derivative

3-1 Introduction to limits

Exercise: Use the graph to fill in the table



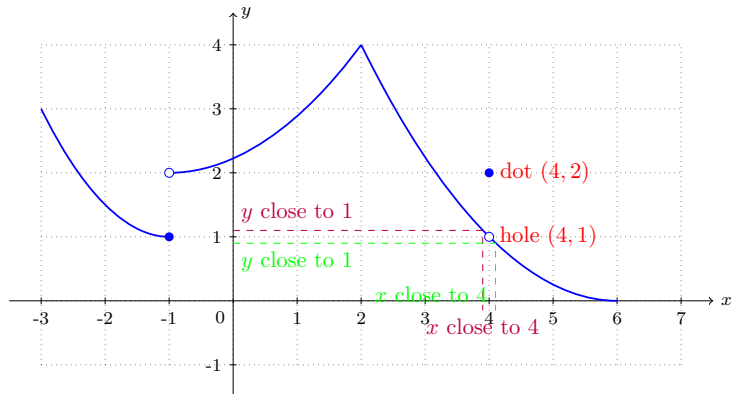
x -value	limit from left	limit from right	limit	y -value
4	$\lim_{x \rightarrow 4^-} f(x) =$	$\lim_{x \rightarrow 4^+} f(x) =$	$\lim_{x \rightarrow 4} f(x) =$	$f(4) =$
-1	$\lim_{x \rightarrow -1^-} f(x) =$	$\lim_{x \rightarrow -1^+} f(x) =$	$\lim_{x \rightarrow -1} f(x) =$	$f(-1) =$



3. Limits and the derivative

3-1 Introduction to limits

Use the graph to fill in the table



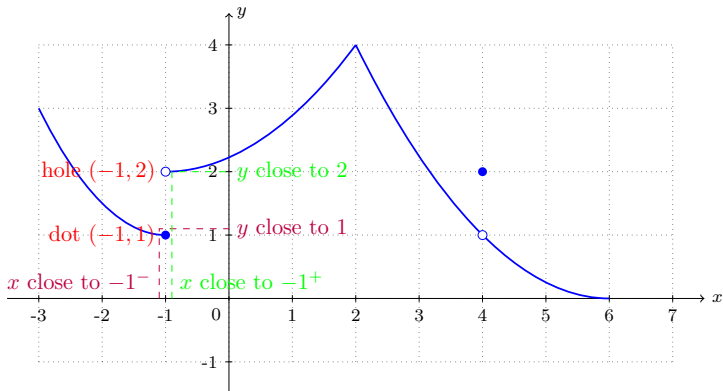
x -value	limit from left	limit from right	limit	y -value
4	$\lim_{x \rightarrow 4^-} f(x) = 1$	$\lim_{x \rightarrow 4^+} f(x) = 1$	$\lim_{x \rightarrow 4} f(x) = 1$	$f(4) = 2$
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3. Limits and the derivative

3-1 Introduction to limits

Use the graph to fill in the table



x -value	limit from left	limit from right	limit	y -value
4	$\lim_{x \rightarrow 4^-} f(x) = 1$	$\lim_{x \rightarrow 4^+} f(x) = 1$	$\lim_{x \rightarrow 4} f(x) = 1$	$f(4) = 2$
-1	$\lim_{x \rightarrow -1^-} f(x) = 1$	$\lim_{x \rightarrow -1^+} f(x) = 2$	$\lim_{x \rightarrow -1} f(x) = DNE$	$f(-1) = 1$

3. Limits and the derivative

3-1 Introduction to limits

DEFINITION One-Sided Limits

If the function passes these three tests:

1 $\lim_{x \rightarrow c^-} f(x)$ exists

2 $\lim_{x \rightarrow c^+} f(x)$ exists

3 Those two limits **match**

Then we say that

$$\lim_{x \rightarrow c} f(x)$$

exists

The value of the limit is whatever was the common value of the left and right limits



3. Limits and the derivative

3-1 Introduction to limits

THEOREM 2 Properties of Limits

Let f and g be two functions, and assume that the following two limits exist and are finite:

$$\lim_{x \rightarrow c} f(x) = L, \quad \lim_{x \rightarrow c} g(x) = M$$

Then

- 1 the limit of a **constant** is the constant
- 2 the limit of x as x approaches c is c
- 3 the limit of the **sum** of the functions is equal to the sum of the limits
- 4 the limit of the **difference** of the functions is equal to the difference of the limits
- 5 the limit of a **constant times a function** is equal to the constant times the limit of the function
- 6 the limit of the **product** of the functions is the product of the limits of the functions
- 7 the limit of the **quotient** of the functions is the quotient of the limits of the functions, provided $M \neq 0$.
- 8 the limit of the n^{th} **root of a function** is the n th root of the limit of that function

3. Limits and the derivative

3-1 Introduction to limits

Example:

Let $f(x) = -7x^2 + 13x - 29$, find $\lim_{x \rightarrow 2} f(x)$



3. Limits and the derivative

3-1 Introduction to limits

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Let $f(x) = -7x^2 + 13x - 29$, find $\lim_{x \rightarrow 2} f(x)$

$$\lim_{x \rightarrow 2} f(x) = \lim_{x \rightarrow 2} -7x^2 + 13x - 29$$



3. Limits and the derivative

3-1 Introduction to limits

Example:

Let $f(x) = -7x^2 + 13x - 29$, find $\lim_{x \rightarrow 2} f(x)$

$$\begin{aligned}\lim_{x \rightarrow 2} f(x) &= \lim_{x \rightarrow 2} -7x^2 + 13x - 29 \\ &= \lim_{x \rightarrow 2} (-7x^2) + \lim_{x \rightarrow 2} (13x) + \lim_{x \rightarrow 2} (-29) \quad \text{Th 2.3}\end{aligned}$$



3. Limits and the derivative

3-1 Introduction to limits

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3. Limits and the derivative

3-1 Introduction to limits

Example:

Let $f(x) = -7x^2 + 13x - 29$, find $\lim_{x \rightarrow 2} f(x)$

$$\begin{aligned}\lim_{x \rightarrow 2} f(x) &= \lim_{x \rightarrow 2} -7x^2 + 13x - 29 \\ &= \lim_{x \rightarrow 2} (-7x^2) + \lim_{x \rightarrow 2} (13x) + \lim_{x \rightarrow 2} (-29) \quad \text{Th 2.3} \\ &= -7 \lim_{x \rightarrow 2} (x^2) + 13 \lim_{x \rightarrow 2} (x) - 29 \quad \text{Th2.5 \& Th2.1} \\ &= -7 \lim_{x \rightarrow 2} (x \times x) + 13 \times 2 - 29 \quad \text{Th2.2}\end{aligned}$$



3. Limits and the derivative

3-1 Introduction to limits

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3. Limits and the derivative

3-1 Introduction to limits

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3-1 Introduction to limits

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3. Limits and the derivative

3-1 Introduction to limits

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3. Limits and the derivative

3-1 Introduction to limits

Example:

Let $f(x) = -7x^2 + 13x - 29$, find $\lim_{x \rightarrow 2} f(x)$

Alternate solution

$$\lim_{x \rightarrow 2} f(x) = \lim_{x \rightarrow 2} -7x^2 + 13x - 29 \quad \text{notice: } f \text{ is a polynomial}$$



3. Limits and the derivative

3-1 Introduction to limits

Example:

Let $f(x) = -7x^2 + 13x - 29$, find $\lim_{x \rightarrow 2} f(x)$

Alternate solution

$$\begin{aligned}\lim_{x \rightarrow 2} f(x) &= \lim_{x \rightarrow 2} -7x^2 + 13x - 29 && \text{notice: } f \text{ is a polynomial} \\ &= -7(2)^2 + 13(2) - 29 && \text{can just substitute in } x = 2 \text{ by using Th3}\end{aligned}$$



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3-1 Introduction to limits

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3-1 Introduction to limits

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From this example, we conclude that

THEOREM 3 Limits of Polynomial and Rational Functions

$$\lim_{x \rightarrow c} f(x) = f(c) \quad \text{for } f \text{ any polynomial function}$$

To find the limit of a polynomial, just plug-in the value!



3. Limits and the derivative

3-1 Introduction to limits

Example:

Let $r(x) = \frac{2x}{3x+1}$, find $\lim_{x \rightarrow 4} r(x)$

$$\lim_{x \rightarrow 4} r(x) = \lim_{x \rightarrow 4} \frac{2x}{3x + 1}$$



3. Limits and the derivative

3-1 Introduction to limits

Example:

Let $r(x) = \frac{2x}{3x+1}$, find $\lim_{x \rightarrow 4} r(x)$

$$\begin{aligned}\lim_{x \rightarrow 4} r(x) &= \lim_{x \rightarrow 4} \frac{2x}{3x+1} \\ &= \frac{\lim_{x \rightarrow 4} 2x}{\lim_{x \rightarrow 4} 3x+1} \quad \text{Th2.7}\end{aligned}$$



3. Limits and the derivative

3-1 Introduction to limits

Example:

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3. Limits and the derivative

3-1 Introduction to limits

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3. Limits and the derivative

3-1 Introduction to limits

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From this example, we conclude that

THEOREM 3 Limits of Polynomial and Rational Functions

$$\lim_{x \rightarrow c} r(x) = r(c) \quad r \text{ any rational function with a nonzero denominator at } x = c$$

3. Limits and the derivative

3-1 Introduction to limits

Indeterminate Forms

It is important to note that there are restrictions on some of the limit properties. In particular if

$$\lim_{x \rightarrow c} f(x) = 0$$

and

$$\lim_{x \rightarrow c} g(x) = 0$$

Then finding

$$\lim_{x \rightarrow c} \frac{f(x)}{g(x)}$$

may present difficulties, since the denominator is 0 (limit property 7 does not apply). In this case, $\lim_{x \rightarrow c} \frac{f(x)}{g(x)}$ is said to be **indeterminate**.

The term "indeterminate" is used because the limit may or may not exist.



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Does this tell us that the limit does not exist ? **NO!**



3. Limits and the derivative

3-1 Introduction to limits

Example: $\frac{0}{0}$

This example illustrates some techniques that can be useful for indeterminate forms. Evaluate the following limit:

$$\lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2} =$$



3. Limits and the derivative

3-1 Introduction to limits

Example: $\frac{0}{0}$

This example illustrates some techniques that can be useful for indeterminate forms. Evaluate the following limit:

$$\lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2} = \lim_{x \rightarrow 2} \frac{(x - 2)(x + 2)}{x - 2}$$



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3-1 Introduction to limits

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This example illustrates some techniques that can be useful for indeterminate forms. Evaluate the following limit:

$$\begin{aligned}\lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2} &= \lim_{x \rightarrow 2} \frac{(x - 2)(x + 2)}{x - 2} \\ &= \lim_{x \rightarrow 2} \frac{\cancel{(x - 2)}(x + 2)}{\cancel{x - 2}} \quad (x \neq 2 \text{ see conceptual insight p135})\end{aligned}$$



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3. Limits and the derivative

3-1 Introduction to limits

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$$\begin{aligned}\lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2} &= \lim_{x \rightarrow 2} \frac{(x - 2)(x + 2)}{x - 2} \\ &= \lim_{x \rightarrow 2} \frac{\cancel{(x - 2)}(x + 2)}{\cancel{x - 2}} \quad (x \neq 2 \text{ see conceptual insight p135}) \\ &= \lim_{x \rightarrow 2} (x + 2) \\ &= 4\end{aligned}$$

Algebraic simplification is often useful when the numerator and denominator are both approaching 0



3. Limits and the derivative

3-1 Introduction to limits

THEOREM 4 Limit of a Quotient

If $\lim_{x \rightarrow c} f(x) = L$, $L \neq 0$, and $\lim_{x \rightarrow c} g(x) = 0$, then

$$\lim_{x \rightarrow c} \frac{f(x)}{g(x)} \text{ does not exist}$$



3. Limits and the derivative

3-1 Introduction to limits

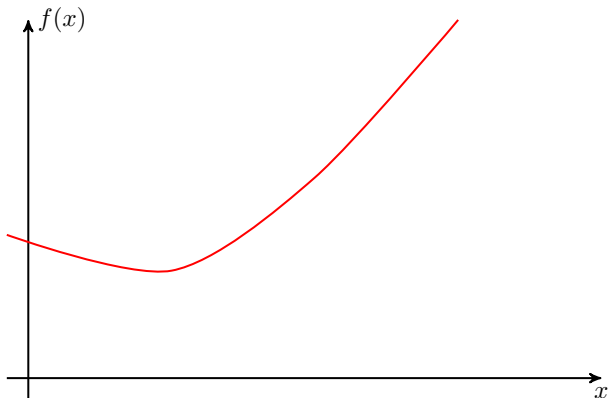
Difference Quotient



3. Limits and the derivative

3-1 Introduction to limits

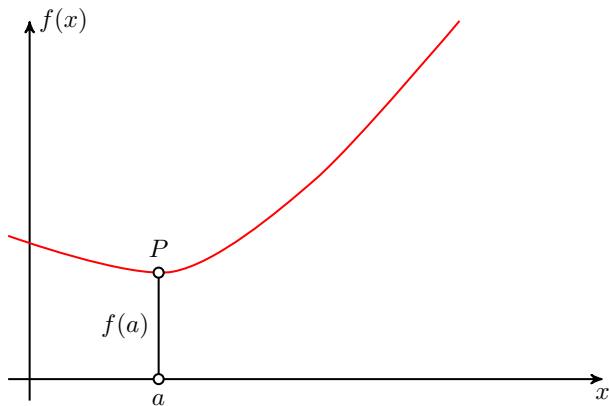
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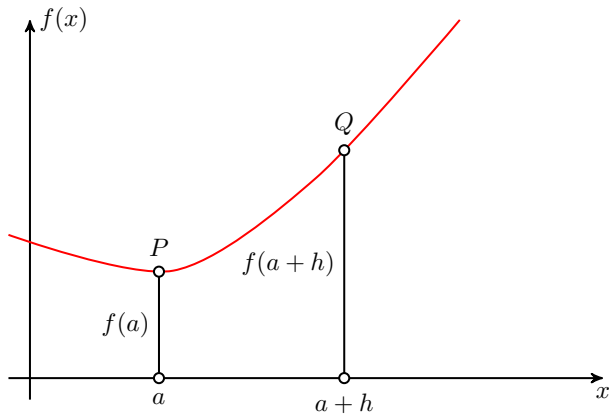
Difference Quotient



3. Limits and the derivative

3-1 Introduction to limits

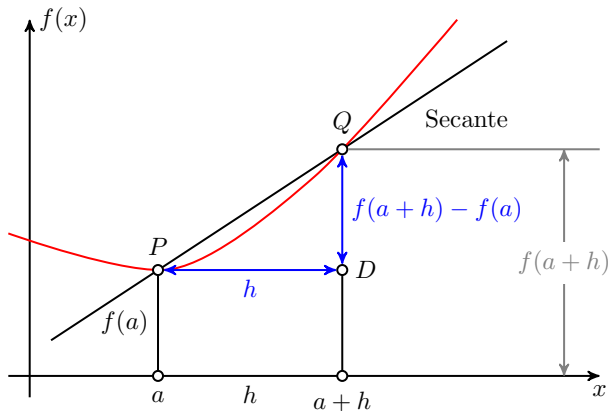
Difference Quotient



3. Limits and the derivative

3-1 Introduction to limits

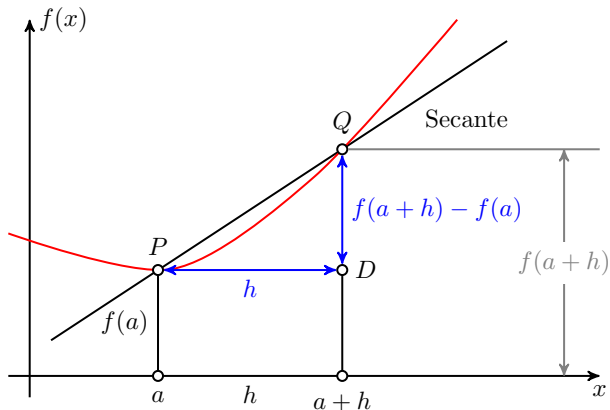
Difference Quotient



3. Limits and the derivative

3-1 Introduction to limits

Difference Quotient



$$\text{Slope of } PQ = \frac{QD}{PD} = \frac{f(a + h) - f(a)}{(a + h) - a} = \frac{f(a + h) - f(a)}{h}$$



3. Limits and the derivative

3-1 Introduction to limits

Example:

Let $f(x) = 3x - 1$, find $\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$

$$f(a + h) =$$



3. Limits and the derivative

3-1 Introduction to limits

Example:

Let $f(x) = 3x - 1$, find $\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$

$$f(a + h) = 3(a + h) - 1$$



3. Limits and the derivative

3-1 Introduction to limits

Example:

Let $f(x) = 3x - 1$, find $\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$

$$f(a+h) = 3(a+h) - 1$$

$$f(a) = 3a - 1$$



3. Limits and the derivative

3-1 Introduction to limits

Example:

Let $f(x) = 3x - 1$, find $\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$

$$f(a+h) = 3(a+h) - 1$$

$$f(a) = 3a - 1$$

$$f(a+h) - f(a) = 3h$$



3. Limits and the derivative

3-1 Introduction to limits

Example:

Let $f(x) = 3x - 1$, find $\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$

$$f(a+h) = 3(a+h) - 1$$

$$f(a) = 3a - 1$$

$$f(a+h) - f(a) = 3h$$

$$\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \rightarrow 0} \frac{3h}{h} = 3$$



3. Limits and the derivative

3-1 Introduction to limits

Exercises:

1. Evaluate the following limits.

a

$$\lim_{x \rightarrow 3} \frac{x - 2}{x - 3}$$

b

$$\lim_{x \rightarrow -3} \frac{x^2 - 9}{x - 3}$$

c

$$\lim_{x \rightarrow 1} \frac{x^2 + 6x - 7}{x - 1}$$

1. Compute the following limits for each function: $\lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h}$

a $f(x) = 4x + 3$

b $f(x) = x^2 - 2x - 5$

c $f(x) = \sqrt{x + 3}$



3. Limits and the derivative

1 3-1 Introduction to limits

2 3-2 Infinite Limits and Limits at Infinity

3 3-3 Continuity

4 3-4 The Derivative

5 3-5 Basic Differentiation Properties

6 3-6 Differentials

7 3-7 Marginal Analysis in Business and Economics



3. Limits and the derivative

3-2 Infinite Limits and Limits at Infinity

Learning Objectives

- Determine infinite limits.
- Locate vertical asymptotes.
- Locate horizontal asymptotes.



3. Limits and the derivative

3-2 Infinite Limits and Limits at Infinity

In section 3.1, we introduced the expression

$$\lim_{x \rightarrow c} f(x) = L$$

Spoken: "The limit as x approaches c of $f(x)$ is L "

Meaning: The graph of f appears to be heading for the location $(x, y) = (c, L)$.

In section 3.2, we will expand our use of the limit symbol, and expand our definition of limit.



3. Limits and the derivative

3-2 Infinite Limits and Limits at Infinity

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In section 3.2, we will expand our use of the limit symbol, and expand our definition of limit.

DEFINITION Infinite Limits and Vertical Asymptotes

The vertical line $x = a$ is a **vertical asymptote** for the graph of $y = f(x)$ if

$$f(x) \rightarrow \infty \quad \text{or} \quad f(x) \rightarrow -\infty \quad \text{as} \quad x \rightarrow a^+ \quad \text{or} \quad x \rightarrow a^-$$

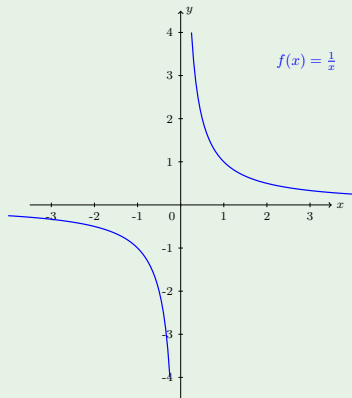
Infinite limits and vertical asymptotes are used to describe the behavior of functions that are unbounded near $x = a$.



3. Limits and the derivative

3-2 Infinite Limits and Limits at Infinity

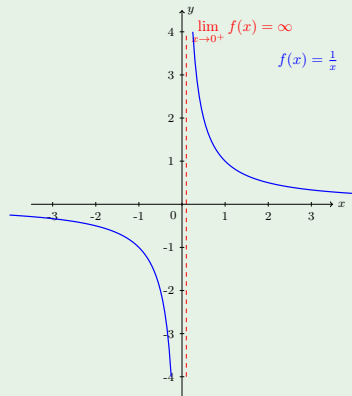
Example



3. Limits and the derivative

3-2 Infinite Limits and Limits at Infinity

Example

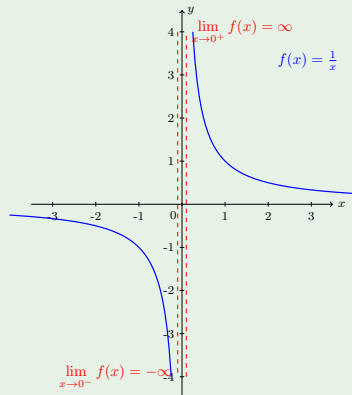


As x gets closer and closer to 0 **from the right** but not equal to 0 the y -values go to ∞ .

3. Limits and the derivative

3-2 Infinite Limits and Limits at Infinity

Example

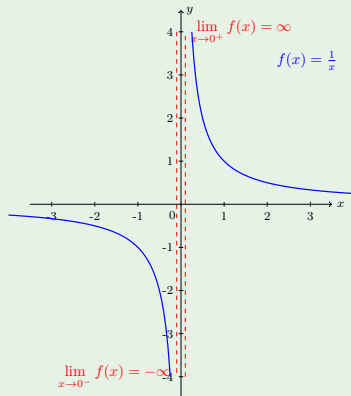


As x gets closer and closer to 0 **from the left** but not equal to 0 the y -values go to $-\infty$.

3. Limits and the derivative

3-2 Infinite Limits and Limits at Infinity

Example



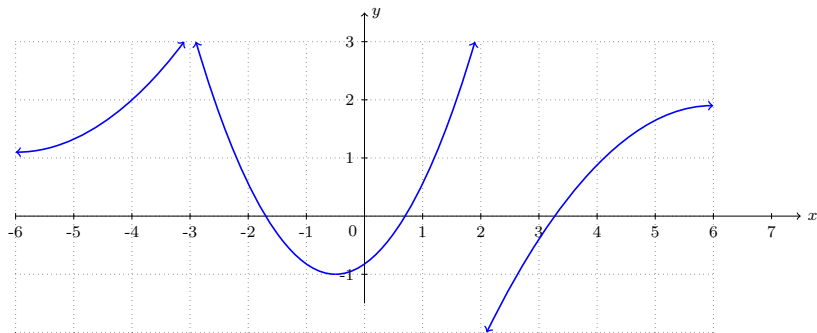
As x gets closer and closer to 0 **from the left** but not equal to 0 the y-values go to $-\infty$.

$$\lim_{x \rightarrow 0^+} f(x) = +\infty, \quad \lim_{x \rightarrow 0^-} f(x) = -\infty, \quad \lim_{x \rightarrow 0} f(x) = DNE$$

3. Limits and the derivative

3-2 Infinite Limits and Limits at Infinity

Exercice: Use the graph to fill in the table



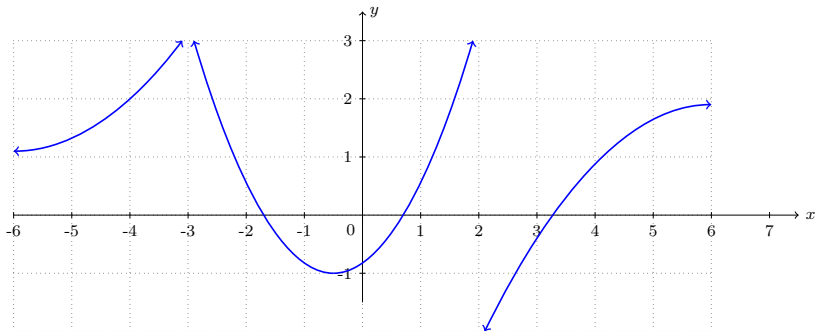
x -value	limit from left	limit from right	limit
-3	$\lim_{x \rightarrow -3^-} f(x) =$	$\lim_{x \rightarrow -3^+} f(x) =$	$\lim_{x \rightarrow -3} f(x) =$
2	$\lim_{x \rightarrow 2^-} f(x) =$	$\lim_{x \rightarrow 2^+} f(x) =$	$\lim_{x \rightarrow 2} f(x) =$



3. Limits and the derivative

3-2 Infinite Limits and Limits at Infinity

Exercise: Use the graph to fill in the table



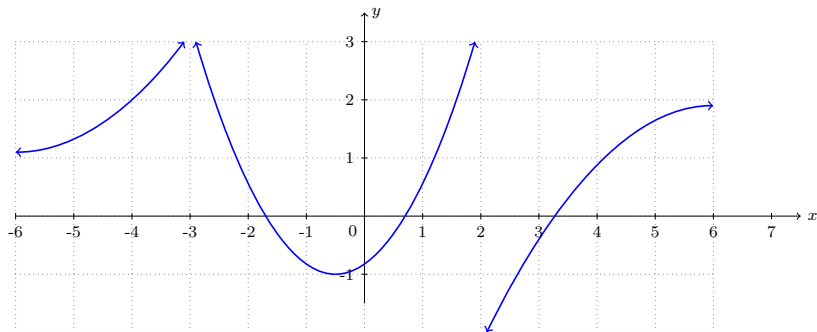
<i>x</i> -value	limit from left	limit from right	limit
-3	$\lim_{x \rightarrow -3^-} f(x) = \infty$	$\lim_{x \rightarrow -3^+} f(x) = \infty$	$\lim_{x \rightarrow -3} f(x) = \infty$
2	$\lim_{x \rightarrow 2^-} f(x) =$	$\lim_{x \rightarrow 2^+} f(x) =$	$\lim_{x \rightarrow 2} f(x) =$



3. Limits and the derivative

3-2 Infinite Limits and Limits at Infinity

Exercise: Use the graph to fill in the table



<i>x</i> -value	limit from left	limit from right	limit
-3	$\lim_{x \rightarrow -3^-} f(x) = \infty$	$\lim_{x \rightarrow -3^+} f(x) = \infty$	$\lim_{x \rightarrow -3} f(x) = \infty$
2	$\lim_{x \rightarrow 2^-} f(x) = \infty$	$\lim_{x \rightarrow 2^+} f(x) = -\infty$	$\lim_{x \rightarrow 2} f(x) = DNE$



3. Limits and the derivative

3-2 Infinite Limits and Limits at Infinity

THEOREM1 Locating Vertical Asymptotes of Rational Functions

If $f(x) = \frac{n(x)}{d(x)}$ is a rational function, $d(c) = 0$ and $n(c) \neq 0$, then the line $x = c$ is a **vertical asymptote** of the graph of f .



3. Limits and the derivative

3-2 Infinite Limits and Limits at Infinity

THEOREM1 Locating Vertical Asymptotes of Rational Functions

If $f(x) = \frac{n(x)}{d(x)}$ is a rational function, $d(c) = 0$ and $n(c) \neq 0$, then the line $x = c$ is a **vertical asymptote** of the graph of f .

In other words

Vertical asymptotes occur for those value of x that produce 0 in the denominator **BUT NOT** in the numerator. (If $\frac{0}{0}$ occurs, you simply have a hole in the graph).



3. Limits and the derivative

3-2 Infinite Limits and Limits at Infinity

Exercise :

Find any vertical asymptotes :

$$a) f(x) = \frac{5}{x^2 - 9}, \quad b) g(x) = \frac{x - 1}{x - 4}, \quad c) h(x) = \frac{x - 2}{x^2 + x - 6}$$



3. Limits and the derivative

3-2 Infinite Limits and Limits at Infinity

Limits at Infinity

Limits at infinity and horizontal asymptotes are used to describe the behavior of functions as x assumes arbitrarily large positive values or arbitrarily large negative values.

We begin by considering power functions of the form x^p and $\frac{1}{x^p}$. If p is a positive real number, then x^p increases as x increases. There is no upper bound on the values of x^p . We indicate this behavior by writing

$$\lim_{x \rightarrow \infty} x^p = \infty$$

Since the reciprocals of very large numbers are very small numbers, it follows that $\frac{1}{x^p}$ approaches 0 as x increases without bound. We indicate this behavior by writing

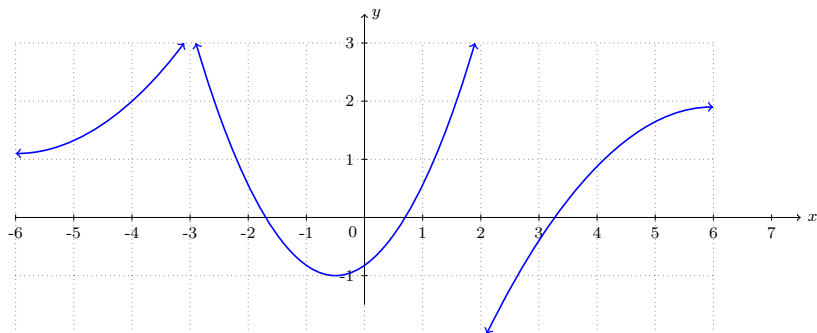
$$\lim_{x \rightarrow \infty} \frac{1}{x^p} = 0$$



3. Limits and the derivative

3-2 Infinite Limits and Limits at Infinity

Exercise: Use the graph to fill in the table



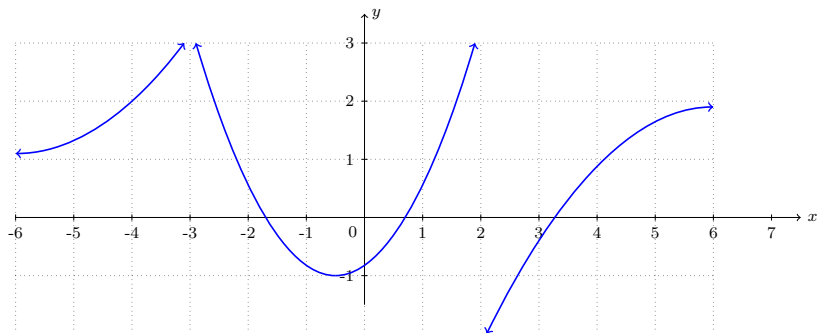
x -value	limit
∞	$\lim_{x \rightarrow \infty} f(x) =$
$-\infty$	$\lim_{x \rightarrow -\infty} f(x) =$



3. Limits and the derivative

3-2 Infinite Limits and Limits at Infinity

Exercise: Use the graph to fill in the table



x -value	limit
∞	$\lim_{x \rightarrow \infty} f(x) = 2$
$-\infty$	$\lim_{x \rightarrow -\infty} f(x) = 1$



3. Limits and the derivative

3-2 Infinite Limits and Limits at Infinity

THEOREM 2 Limits of Power Functions at Infinity

If p is a positive real number and k is any real number except 0, then

1

$$\lim_{x \rightarrow -\infty} \frac{k}{x^p} = 0$$

2

$$\lim_{x \rightarrow \infty} \frac{k}{x^p} = 0$$

3

$$\lim_{x \rightarrow -\infty} kx^p = \pm\infty$$

4

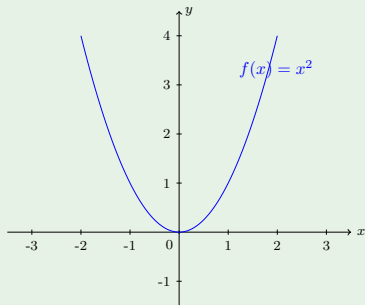
$$\lim_{x \rightarrow \infty} kx^p = \pm\infty$$



3. Limits and the derivative

3-2 Infinite Limits and Limits at Infinity

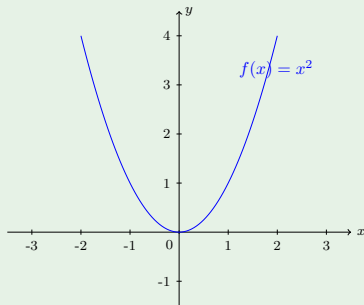
Example: $f(x) = x^2$



3. Limits and the derivative

3-2 Infinite Limits and Limits at Infinity

Example: $f(x) = x^2$



$k = 1 > 0, p = 2$ (even)

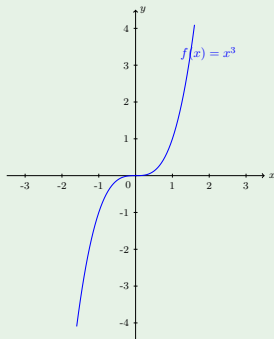
$$\lim_{x \rightarrow +\infty} x^2 = \infty, \quad \lim_{x \rightarrow -\infty} x^2 = \infty$$



3. Limits and the derivative

3-2 Infinite Limits and Limits at Infinity

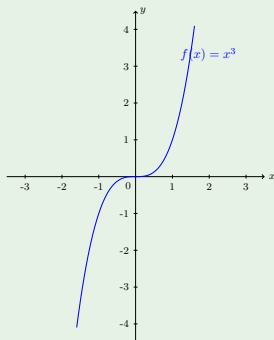
Example: $f(x) = x^3$



3. Limits and the derivative

3-2 Infinite Limits and Limits at Infinity

Example: $f(x) = x^3$



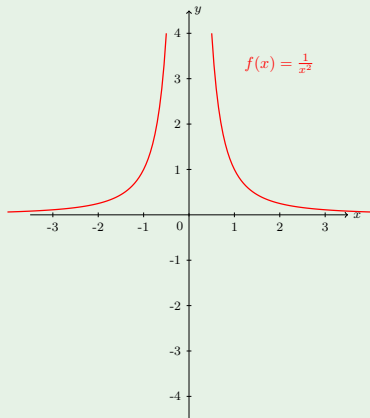
$k = 1 > 0, p = 3$ (odd)

$$\lim_{x \rightarrow +\infty} x^3 = \infty, \quad \lim_{x \rightarrow -\infty} x^3 = -\infty$$

3. Limits and the derivative

3-2 Infinite Limits and Limits at Infinity

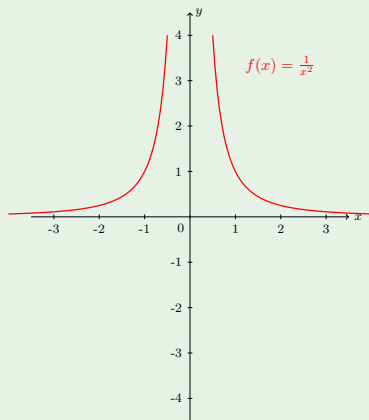
Example: $f(x) = x^{-2}$



3. Limits and the derivative

3-2 Infinite Limits and Limits at Infinity

Example: $f(x) = x^{-2}$



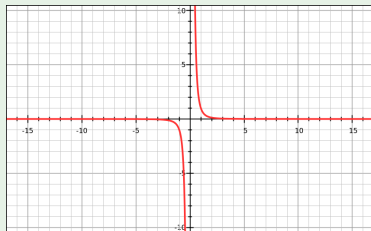
$k = 1 > 0, p = 2$ (even)

$$\lim_{x \rightarrow +\infty} \frac{1}{x^2} = 0, \quad \lim_{x \rightarrow -\infty} \frac{1}{x^2} = 0$$

3. Limits and the derivative

3-2 Infinite Limits and Limits at Infinity

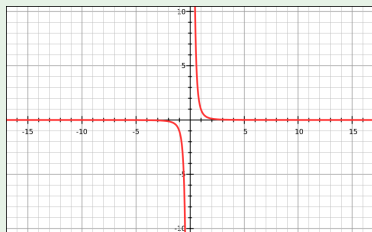
Example: $f(x) = x^{-3}$



3. Limits and the derivative

3-2 Infinite Limits and Limits at Infinity

Example: $f(x) = x^{-3}$



$k = 1 > 0, p = 3$ (odd)

$$\lim_{x \rightarrow +\infty} \frac{1}{x^3} = 0, \quad \lim_{x \rightarrow -\infty} \frac{1}{x^3} = 0$$



3. Limits and the derivative

3-2 Infinite Limits and Limits at Infinity

THEOREM 3 Limits of Polynomial Functions at Infinity

If

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x^1 + a_0, \quad a_n \neq 0, n \geq 1$$

then

$$\lim_{x \rightarrow \infty} p(x) = \lim_{x \rightarrow \infty} a_n x^n = \pm \infty$$

and

$$\lim_{x \rightarrow -\infty} p(x) = \lim_{x \rightarrow -\infty} a_n x^n = \pm \infty$$

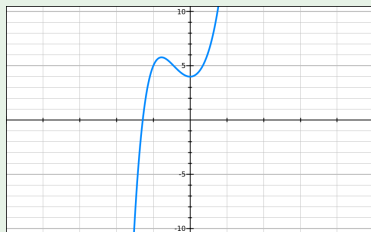
Each limit will be either $-\infty$ or ∞ , depending on a_n and n .



3. Limits and the derivative

3-2 Infinite Limits and Limits at Infinity

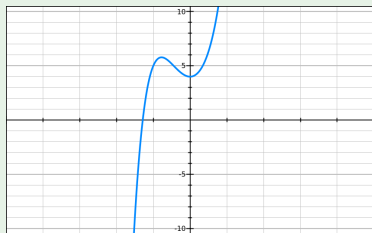
Example: $p(x) = 2x^5 + 4x^3 + 7x^2 + 4$



3. Limits and the derivative

3-2 Infinite Limits and Limits at Infinity

Example: $p(x) = 2x^5 + 4x^3 + 7x^2 + 4$



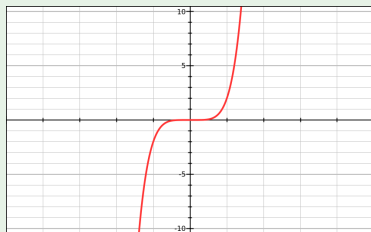
$$\lim_{x \rightarrow +\infty} p(x) = \lim_{x \rightarrow +\infty} 2x^5 + 4x^3 + 7x^2 + 4 = \infty$$



3. Limits and the derivative

3-2 Infinite Limits and Limits at Infinity

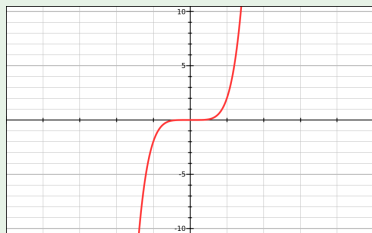
Example: $q(x) = 2x^5$



3. Limits and the derivative

3-2 Infinite Limits and Limits at Infinity

Example: $q(x) = 2x^5$



$$\lim_{x \rightarrow +\infty} q(x) = \lim_{x \rightarrow +\infty} 2x^5 = \infty$$



3. Limits and the derivative

3-2 Infinite Limits and Limits at Infinity

THEOREM 4 Limits of Rational Functions at Infinity and Horizontal Asymptotes of Rational Functions

1 If

$$f(x) = \frac{a_m x^m + a_{m-1} x^{m-1} + \dots + a_1 x^1 + a_0}{b_n x^n + b_{n-1} x^{n-1} + \dots + b_1 x^1 + b_0}, \quad a_m \neq 0, b_n \neq 0$$

then

$$\lim_{x \rightarrow \infty} f(x) = \frac{a_m x^m}{b_n x^n}, \quad \text{and} \quad \lim_{x \rightarrow -\infty} f(x) = \frac{a_m x^m}{b_n x^n}$$

2 There are three possible cases for these limits:

- a) if $m < n$, then $\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow -\infty} f(x) = 0$ and the line $y = 0$ (the x axis) is a horizontal asymptote of $f(x)$.
- b) if $m = n$, then $\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow -\infty} f(x) = \frac{a_m}{b_n}$ and the line $y = \frac{a_m}{b_n}$ is a horizontal asymptote of $f(x)$.
- c) if $m > n$, then each limit will be $\pm\infty$, depending on m , n , a_m and b_n , and $f(x)$ does not have a horizontal asymptote



3. Limits and the derivative

3-2 Infinite Limits and Limits at Infinity

Exercises:

1. Find each limit for $f(x) = \frac{x-1}{x+2}$.

a $\lim_{x \rightarrow -2^+} f(x)$

b $\lim_{x \rightarrow -2^-} f(x)$

c $\lim_{x \rightarrow -2} f(x)$

1. Evaluate the indicated limit

a $\lim_{x \rightarrow \infty} \frac{x+3}{2x-1}$

b $\lim_{x \rightarrow \infty} \frac{4x^3+2x}{x^2-1}$

3. Discuss three different functions f, g, h

$$a) f(x) = \frac{9x^2 - 90x + 189}{2x^2 - 24x + 70}, \quad b) g(x) = \frac{9x^2 - 90x + 189}{2x^3 - 24x^2 + 70x}, \quad c) h(x) = \frac{9x^3 - 90x^2 + 189x}{2x^2 - 24x + 70}$$

a Would the graph of f, g, h have horizontal asymptotes?

b How can we find out without drawing the graph?

c Answer by taking the limits as $x \rightarrow \infty$ or $x \rightarrow -\infty$.

3. Limits and the derivative

1 3-1 Introduction to limits

2 3-2 Infinite Limits and Limits at Infinity

3 3-3 Continuity

4 3-4 The Derivative

5 3-5 Basic Differentiation Properties

6 3-6 Differentials

7 3-7 Marginal Analysis in Business and Economics



3. Limits and the derivative

3-3 Continuity

Learning Objectives

- Use the definition of continuity to determine if a function is continuous.
- Use continuity properties to determine intervals of continuity for symbolic functions.
- Construct sign charts to solve inequalities.



3. Limits and the derivative

3-3 Continuity

DEFINITION Continuity

A function f is continuous at the point $x = c$ if

1 $\lim_{x \rightarrow c} f(x)$ **exists**

2 $f(c)$ **exists**

3 $\lim_{x \rightarrow c} f(x) = f(c)$

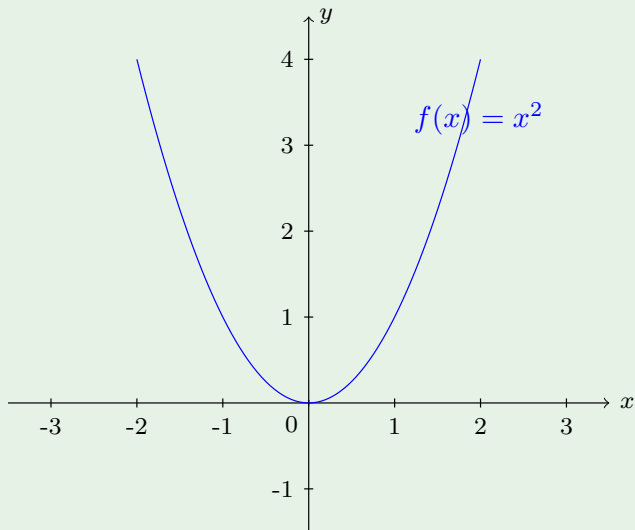
A function is continuous on the open interval (a, b) if it is continuous at each point on the interval.



3. Limits and the derivative

3-3 Continuity

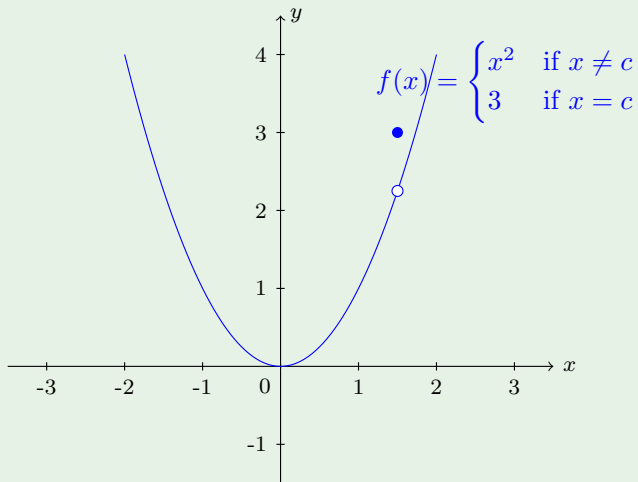
Example:



3. Limits and the derivative

3-3 Continuity

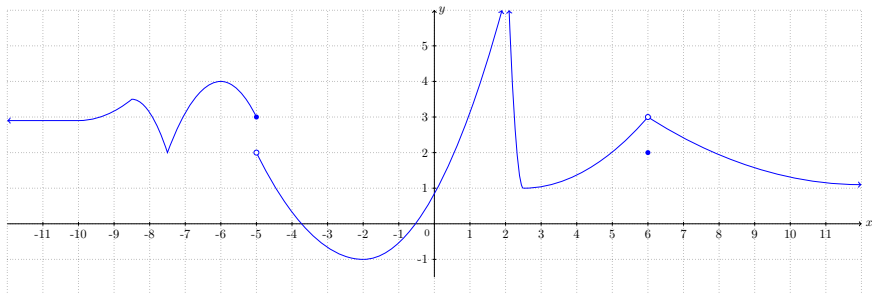
Example:



3. Limits and the derivative

3-3 Continuity

Exercise: Use the graph to answer the questions that follow



1 For each asymptote, give the line equation and say whether it is horizontal/vertical.

2 $\lim_{x \rightarrow -\infty} f(x)$, $\lim_{x \rightarrow -5} f(x)$, $\lim_{x \rightarrow -2} f(x)$, $\lim_{x \rightarrow 6} f(x)$, $\lim_{x \rightarrow \infty} f(x)$

3 If f continuous at $a = -5$? If not, explain why not.

4 If f continuous at $a = -2$? If not, explain why not.

5 If f continuous at $a = 2$? If not, explain why not.

6 If f continuous at $a = 6$? If not, explain why not.



3. Limits and the derivative

3-3 Continuity

Exercise : Discuss the continuity of each function at the indicated point

1

$$f(x) = x + 1, \quad \text{at } x = 10$$

2

$$g(x) = \frac{x^2 - 9}{x - 3}, \quad \text{at } x = 3$$

3

$$h(x) = \frac{|x - 1|}{x - 1}, \quad \text{at } x = 1, \quad \text{and } x = 0$$



3. Limits and the derivative

3-3 Continuity

If two functions are continuous on the same interval, then their sum, difference, product, and quotient are continuous on the same interval except for values of x that make a denominator 0.

THEOREM 1 Continuity Properties of Some Specific Functions

- 1 A constant function $f(x) = k$, where k is a constant, is continuous for all x .
 $f(x) = 2$ is continuous for all x .
- 2 For n a positive integer, $f(x) = x^n$ is continuous for all x .
 $f(x) = x^{22}$ is continuous for all x .
- 3 A polynomial function is continuous for all x .
 $f(x) = 3x^3 + 2x^2 - 2$ is continuous for all x .
- 4 A rational function is continuous for all x except those values that make a denominator 0.
 $\frac{x^2+1}{x-4}$ is continuous for all x except $x = 4$, a value that makes the denominator 0.
- 5 For n an odd positive integer greater than 1, $\sqrt[n]{f(x)}$ is continuous wherever $f(x)$ is continuous.
 $\sqrt[3]{x^2}$ is continuous for all x .
- 6 For n an even positive integer, $\sqrt[n]{f(x)}$ is continuous wherever $f(x)$ is continuous and non negative.
 $\sqrt[4]{x}$ is continuous on the interval $[0, \infty)$.

3. Limits and the derivative

3-3 Continuity

Exercise : Determine where each function is continuous

1

$$f(x) = x^{1024} + 3x^2 + 1$$

2

$$g(x) = \frac{x^2}{(x+1)(x-8)(x+3)}$$

3

$$h(x) = \sqrt[5]{x^2 - 1}$$

4

$$l(x) = \sqrt{x - 4}$$



3. Limits and the derivative

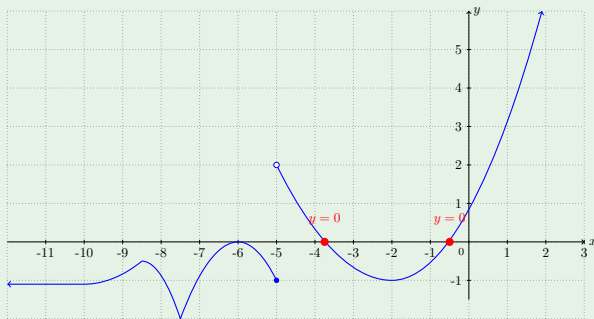
3-3 Continuity

Remark : Notice important behavior of functions

Common question: for some function $f(x)$,

- which x -values will have positive y -values?
- which x -values will have negative y -values?
- which x -values will have $y = 0$?

Observation: The sign of a function f can only change at x -values such that $f(x) = 0$ or f is discontinuous at x .



3. Limits and the derivative

3-3 Continuity

In general, if f is continuous and $f(x) \neq 0$ on the interval (a, b) , then $f(x)$ cannot change sign on (a, b) .

THEOREM 2 Sign Properties on an Interval (a, b)

If f is continuous on (a, b) and for all x in (a, b) , then

- either $f(x) > 0$ for all x in (a, b)
- or $f(x) < 0$ for all x in (a, b) .

Theorem 2 provides the basis for an effective method of solving many types of inequalities.



3. Limits and the derivative

3-3 Continuity

PROCEDURE **Constructing Sign Charts**

Given a function f ,

Step 1 Find all partition numbers:

- 1 Find all numbers such that f is discontinuous.
- 2 Find all numbers such that $f(x) = 0$.

Step 2 Plot the numbers found in step 1 on a real-number line, dividing the number line into intervals.

Step 3 Select a test number in each open interval determined in step 2 and evaluate $f(x)$ at each test number to determine whether $f(x)$ is positive or negative in each interval.

Step 4 Construct a sign chart, using the real-number line in step 2. This will show the sign of $f(x)$ on each open interval.



3. Limits and the derivative

3-3 Continuity

Exercise:

1. Solve

$$\frac{x + 1}{x - 2} > 0$$



3. Limits and the derivative

1 3-1 Introduction to limits

2 3-2 Infinite Limits and Limits at Infinity

3 3-3 Continuity

4 3-4 The Derivative

5 3-5 Basic Differentiation Properties

6 3-6 Differentials

7 3-7 Marginal Analysis in Business and Economics



3. Limits and the derivative

3-4 The Derivative

Learning Objectives

- Interpret the meaning of rate of change in the context of applications.
- Find the derivative using the difference quotient.
- Identify locations of nonexistence of the derivative.



3. Limits and the derivative

3-4 The Derivative

DEFINITION Average Rate of Change

For $y = f(x)$, the **average rate of change** from $x = a$ to $x = a + h$ is

$$\frac{f(a+h) - f(a)}{(a+h) - a} = \frac{f(a+h) - f(a)}{h}, \quad h \neq 0$$

The above expression is also called a **difference quotient**.

It can be interpreted as the **slope of a secant**.

See the picture on the next slide for illustration.



3. Limits and the derivative

3-4 The Derivative

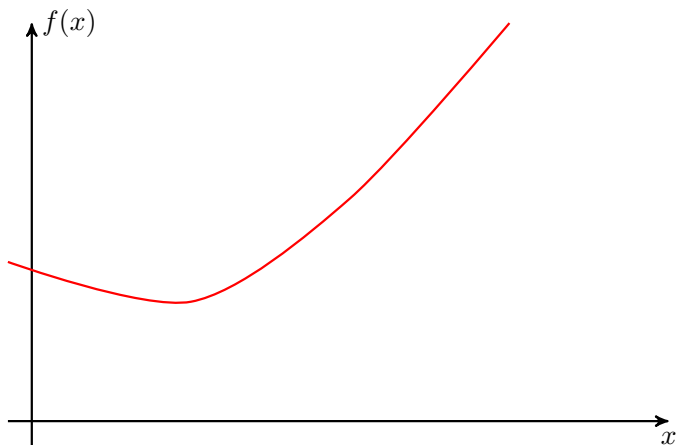
Average Rate of Change, difference quotient, slope of a secant



3. Limits and the derivative

3-4 The Derivative

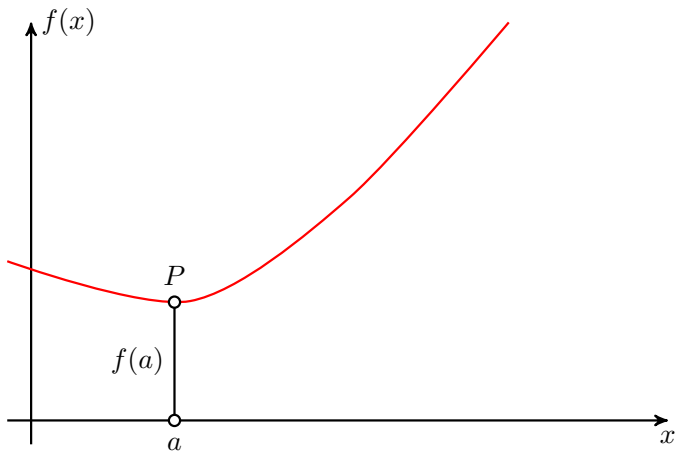
Average Rate of Change, difference quotient, slope of a secant



3. Limits and the derivative

3-4 The Derivative

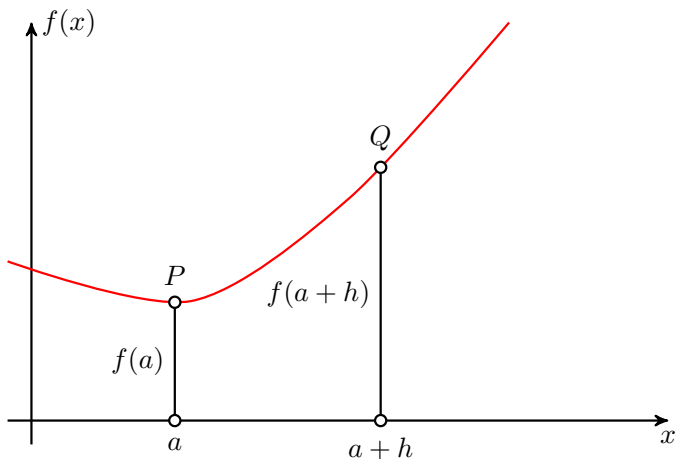
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3. Limits and the derivative

3-4 The Derivative

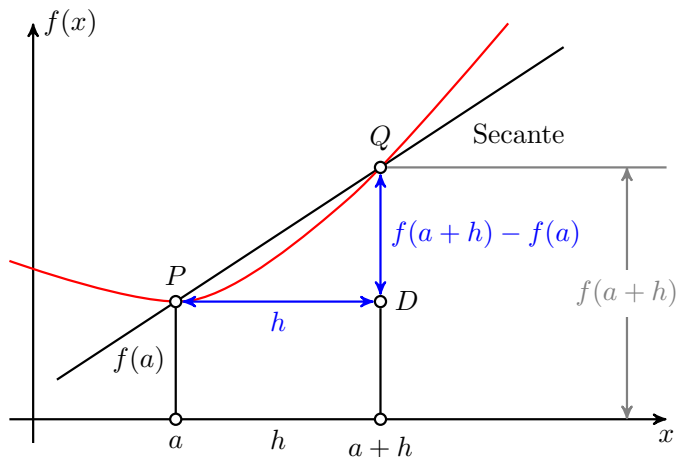
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3. Limits and the derivative

3-4 The Derivative

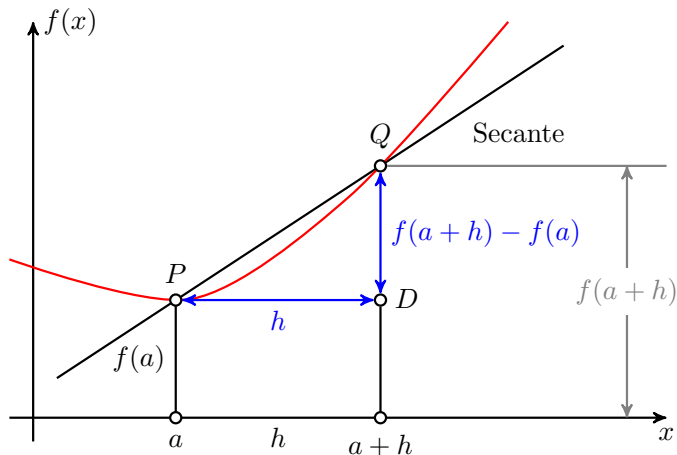
Average Rate of Change, difference quotient, slope of a secant



3. Limits and the derivative

3-4 The Derivative

Average Rate of Change, difference quotient, slope of a secant



$$\text{Slope of } PQ = \frac{QD}{PD} = \frac{f(a+h) - f(a)}{(a+h) - a}$$



3. Limits and the derivative

3-4 The Derivative

Example 1:

The revenue generated by producing and selling widgets is given by

$$R(x) = x(75 - 3x) \quad \text{for } 0 \leq x \leq 20$$

What is the change in revenue if production changes from 9 to 12?



3. Limits and the derivative

3-4 The Derivative

Example 1:

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What is the change in revenue if production changes from 9 to 12?

$$R(12) - R(9) = 12(75 - 3 \cdot 12) - 9(75 - 3 \cdot 9) = \$468 - \$432 = \$36$$

Increasing production from 9 to 12 will increase revenue by \$36.



3. Limits and the derivative

3-4 The Derivative

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3. Limits and the derivative

3-4 The Derivative

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Increasing production from 9 to 12 will increase revenue by \$36.

What is the average rate of change in revenue (per unit change in x) if production changes from 9 to 12?

To find the average rate of change we divide the change in revenue by the change in production:

$$\frac{R(12) - R(9)}{12 - 9} = \frac{36}{3} = 12$$

Thus the average change in revenue is \$12 when production is increased from 9 to 12.



3. Limits and the derivative

3-4 The Derivative

DEFINITION The instantaneous Rate of Change

Consider the function $y = f(x)$ only near the point $P = (a, f(a))$.

The difference quotient

$$\frac{f(a+h) - f(a)}{h}$$

gives the average rate of change of f over the interval $[a, a+h]$.

If we make h smaller and smaller, in the limit we obtain the **instantaneous rate of change** of the function at the point P (at $x = a$):

$$\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

(provides that the limit exists)



3. Limits and the derivative

3-4 The Derivative

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$$\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

(provides that the limit exists)

It can be interpreted as the **slope of the tangent** at the point $P (a, f(a))$.

See illustration on the next slide.



3. Limits and the derivative

3-4 The Derivative



3. Limits and the derivative

3-4 The Derivative

DEFINITION Derivative

For $y = f(x)$, we define the derivative of f at x , denoted $f'(x)$ to be

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

if the limit exists.

- If $f'(a)$ exists, we call f **differentiable at a** .
- If $f'(x)$ exists for each x in the open interval (a, b) , then f is said to be **differentiable over (a, b)** .



3. Limits and the derivative

3-4 The Derivative

Interpretation of Derivative

If f is a function, then f' is a new function with the following interpretations:

- For each x in the domain of f' , $f'(x)$ is **the slope of the line tangent** to the graph of f at the point $(x, f(x))$.
- For each x in the domain of f' , $f'(x)$ is **the instantaneous Rate of Change** of $y = f(x)$ with respect to x .
- If $f(x)$ is the position of a moving object at time x , then $v = f'(x)$ is **the velocity** of the object at that time.



3. Limits and the derivative

3-4 The Derivative

PROCEDURE Finding the Derivative

To find $f'(x)$, we use a four-step process:

Step 1 find $f(x + h)$

Step 2 find $f(x + h) - f(x)$

Step 3 find $\frac{f(x + h) - f(x)}{h}$

Step 4 find $\lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h}$



3. Limits and the derivative

3-4 The Derivative

Example 2:

Find the derivative of $f(x) = x^2 - 3x$



3. Limits and the derivative

3-4 The Derivative

Example 2:

Find the derivative of $f(x) = x^2 - 3x$

$$\text{Step 1 } f(x + h) = (x + h)^2 - 3(x + h) = x^2 + 2hx + h^2 - 3x - 3h$$



3. Limits and the derivative

3-4 The Derivative

Example 2:

Find the derivative of $f(x) = x^2 - 3x$

$$\text{Step 1 } f(x+h) = (x+h)^2 - 3(x+h) = x^2 + 2hx + h^2 - 3x - 3h$$

$$\text{Step 2 } f(x+h) - f(x) = \cancel{x^2} + 2hx + h^2 - \cancel{3x} - 3h - \cancel{x^2} + \cancel{3x} = 2hx + h^2 - 3h$$



3. Limits and the derivative

3-4 The Derivative

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$$\text{Step 3 } \frac{f(x+h) - f(x)}{h} = \frac{2x\cancel{h} + h^{\cancel{2}} - 3\cancel{h}}{\cancel{h}} = 2x + h - 3$$



3. Limits and the derivative

3-4 The Derivative

Example 2:

Find the derivative of $f(x) = x^2 - 3x$

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3. Limits and the derivative

3-4 The Derivative

Example 3:

Find the slope of the tangent to the graph of $f(x) = x^2 - 3x$ at $x = 0$, $x = 2$ and $x = 3$.



3. Limits and the derivative

3-4 The Derivative

Example 3:

Find the slope of the tangent to the graph of $f(x) = x^2 - 3x$ at $x = 0$, $x = 2$ and $x = 3$.

In Example 2, we found the derivative of this function at x to be $f'(x) = 2x - 3$.
Hence,

$$f'(0) = -3$$

$$f'(2) = 1$$

$$f'(3) = 3$$



3. Limits and the derivative

3-4 The Derivative

Exercises:

Find the derivative of $f(x) = 2x - 3x^2$ using the four step process.



3. Limits and the derivative

3-4 The Derivative

Exercises:

Find the derivative of $f(x) = 2x - 3x^2$ using the four step process.

Step 1 $f(x + h) = 2(x + h) - 3(x + h)^2$



3. Limits and the derivative

3-4 The Derivative

Exercises:

Find the derivative of $f(x) = 2x - 3x^2$ using the four step process.

$$\text{Step 1 } f(x + h) = 2(x + h) - 3(x + h)^2$$

$$\text{Step 2 } f(x + h) - f(x) = 2h - 6xh - 3h^2$$



3. Limits and the derivative

3-4 The Derivative

Exercises:

Find the derivative of $f(x) = 2x - 3x^2$ using the four step process.

$$\text{Step 1 } f(x + h) = 2(x + h) - 3(x + h)^2$$

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$$\text{Step 3 } \frac{f(x + h) - f(x)}{h} = \frac{2h - 6xh - 3h^2}{h} = 2 - 6x - 3h$$



3. Limits and the derivative

3-4 The Derivative

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Find the derivative of $f(x) = 2x - 3x^2$ using the four step process.

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3. Limits and the derivative

3-4 The Derivative

Non existence of the Derivative

The existence of a derivative at $x = a$ depends on the existence of the limit

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

If the limit DNE, we say that the function is **nondifferentiable** at $x = a$, or **$f'(a)$ DNE**.



3. Limits and the derivative

3-4 The Derivative

Non existence of the Derivative

The existence of a derivative at $x = a$ depends on the existence of the limit

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

If the limit DNE, we say that the function is **nondifferentiable** at $x = a$, or **$f'(a)$ DNE**. Some of the reasons why the derivative of a function may not exist at $x = a$ are

- The graph of f has a hole or break at $x = a$, or
- The graph of f has a sharp corner at $x = a$, or
- The graph of f has a vertical tangent at $x = a$.



3. Limits and the derivative

3-4 The Derivative

Exercises :

1. Suppose a manufacturer's monthly profit (in dollars) from the sale of x bags of a particular Bermuda grass fertilizer is given by $P(x) = -0.07x^2 + 70x - 100$, where $2 \leq x \leq 998$.
 - a. Find the average rate of change of profit if production is changed from 100 bags of fertilizer monthly to 500 bags of fertilizer monthly.
 - b. Explain the meaning of the value obtained in part **a** in the context of the problem.
 - c. Find the instantaneous rate of change of profit when 200 bags of fertilizer are sold. Explain the meaning of this value in the context of the problem.
2. Use the four-step process to find $f'(x)$ for $f(x) = x + 3x^2$.



3. Limits and the derivative

1 3-1 Introduction to limits

2 3-2 Infinite Limits and Limits at Infinity

3 3-3 Continuity

4 3-4 The Derivative

5 3-5 Basic Differentiation Properties

6 3-6 Differentials

7 3-7 Marginal Analysis in Business and Economics



3. Limits and the derivative

3-5 Basic Differentiation Properties

Learning Objectives

- Calculate the derivative of a constant function.
- Apply the power rule.
- Apply the constant multiple and sum and difference properties.



3. Limits and the derivative

3-5 Basic Differentiation Properties

NOTATION **The Derivative**

Notation for the derivative of a function, if $y = f(x)$, then

without variable f' , y' , $\frac{dy}{dx}$

with variable $f'(x)$, $\frac{d}{dx}f(x)$

All represent the derivative of f at x .



3. Limits and the derivative

3-5 Basic Differentiation Properties

What is the slope of a constant function?



3. Limits and the derivative

3-5 Basic Differentiation Properties

What is the slope of a constant function?

Consider graph of $f(x) = 1$ and consider slopes of tangent lines.

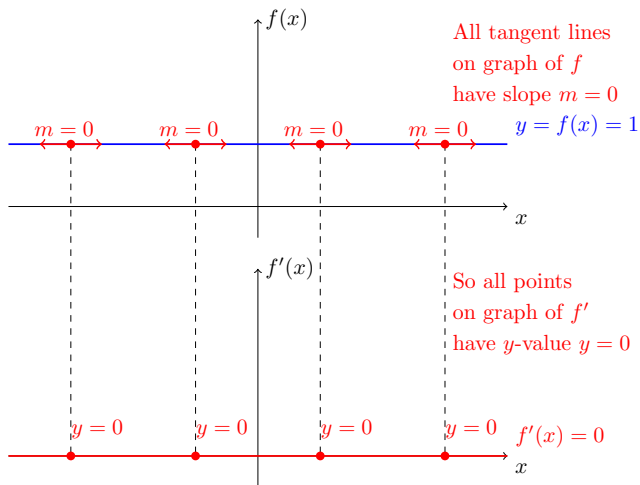


3. Limits and the derivative

3-5 Basic Differentiation Properties

What is the slope of a constant function?

Consider graph of $f(x) = 1$ and consider slopes of tangent lines.



3. Limits and the derivative

3-5 Basic Differentiation Properties

THEOREM 1 Constant Function Rule

Let $y = f(x) = C$ be a constant function, then

$$y' = f'(x) = 0$$



3. Limits and the derivative

3-5 Basic Differentiation Properties

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Let $y = f(x) = C$ be a constant function, then

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Power function

A **power function** is a function of the form

$$f(x) = x^n, \quad n \in \mathbb{R}$$



3. Limits and the derivative

3-5 Basic Differentiation Properties

THEOREM 1 Constant Function Rule

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Power function

A **power function** is a function of the form

$$f(x) = x^n, \quad n \in \mathbb{R}$$

THEOREM 2 The Power Rule (IT WILL BE USED A LOT!)

If $f(x) = x^n$, then

$$f'(x) = nx^{n-1}$$



3. Limits and the derivative

3-5 Basic Differentiation Properties

Example 1:

$f(x) = x^3$, find $f'(x)$.



3. Limits and the derivative

3-5 Basic Differentiation Properties

Example 1:

$$f(x) = x^3, \text{ find } f'(x).$$

By the power rule, the derivative of x^n is nx^{n-1} . In our case $n = 3$, so we get

$$f'(x) = 3x^2.$$



3. Limits and the derivative

3-5 Basic Differentiation Properties

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Example 2:

$$f(x) = \frac{1}{x^3}, \text{ find } f'(x).$$



3. Limits and the derivative

3-5 Basic Differentiation Properties

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3. Limits and the derivative

3-5 Basic Differentiation Properties

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If $f(x) = \frac{1}{x^3}$, then $f'(x) = \frac{1}{3x^2}$ **Invalid!** $\frac{1}{x^3}$ is not written in the form x^n



3. Limits and the derivative

3-5 Basic Differentiation Properties

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Rewrite $f(x) = \frac{1}{x^3} = x^{-3}$, so $f'(x) = -3x^{-3-1} = -3x^{-4} = \frac{-3}{x^4}$



3. Limits and the derivative

3-5 Basic Differentiation Properties

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$$f(x) = x^\pi, \text{ find } f'(x).$$



3. Limits and the derivative

3-5 Basic Differentiation Properties

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3. Limits and the derivative

3-5 Basic Differentiation Properties

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3. Limits and the derivative

3-5 Basic Differentiation Properties

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By the power rule, the derivative of x^n is nx^{n-1} . In our case $n = \pi$, so we get

$$f'(x) = \pi x^{\pi-1}.$$

Example 4:

$$f(x) = 3^x, \text{ find } f'(x).$$

Not a power function. Power rule does not apply!! Can't do it (this week)

3. Limits and the derivative

3-5 Basic Differentiation Properties

Example 5:

$f(x) = 3^5$, find $f'(x)$.



3. Limits and the derivative

3-5 Basic Differentiation Properties

Example 5:

$$f(x) = 3^5, \text{ find } f'(x).$$

Constant function, $f'(x) = 0$ (not $5 \cdot 3^4$)



3. Limits and the derivative

3-5 Basic Differentiation Properties

Example 5:

$$f(x) = 3^5, \text{ find } f'(x).$$

Constant function, $f'(x) = 0$ (not $5 \cdot 3^4$)

Example 6:

$$f(x) = \sqrt[5]{x}, \text{ find } f'(x).$$



3. Limits and the derivative

3-5 Basic Differentiation Properties

Example 5:

$$f(x) = 3^5, \text{ find } f'(x).$$

Constant function, $f'(x) = 0$ (not $5 \cdot 3^4$)

Example 6:

$$f(x) = \sqrt[5]{x}, \text{ find } f'(x).$$

Must rewrite f as a power function $f(x) = \sqrt[5]{x} = x^{\frac{1}{5}}$.

$$f'(x) = \left(\frac{1}{5}\right) x^{\frac{1}{5}-1} = \left(\frac{1}{5}\right) x^{\frac{-4}{5}} = \frac{1}{5x^{\frac{4}{5}}}$$



3. Limits and the derivative

3-5 Basic Differentiation Properties

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$$f(x) = x, \text{ find } f'(x).$$



3. Limits and the derivative

3-5 Basic Differentiation Properties

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Example 7:

$$f(x) = x, \text{ find } f'(x).$$

Write f as a power function $f(x) = x^1$.

$$f'(x) = 1x^{1-1} = 1 \cdot x^0 = 1 \cdot 1 = 1 \quad \left(\frac{dx}{dx} = 1\right)$$



3. Limits and the derivative

3-5 Basic Differentiation Properties

Does this make sense graphically?

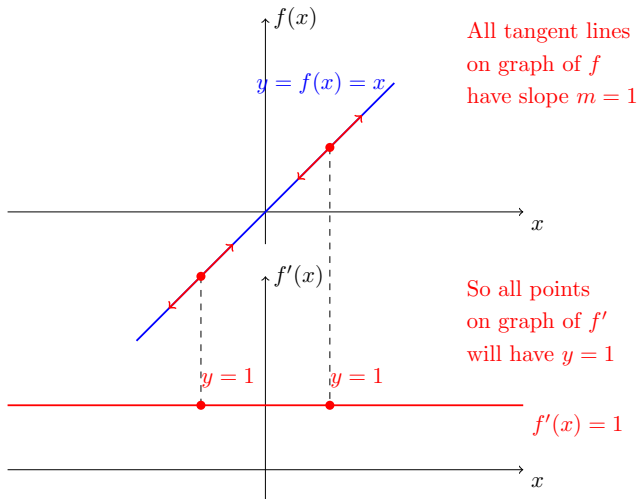


3. Limits and the derivative

3-5 Basic Differentiation Properties

Does this make sense graphically?

Consider graph of $f(x) = x$ and find $f'(x)$ graphically.



3. Limits and the derivative

3-5 Basic Differentiation Properties

THEOREM 3 Constant Multiple Property

If $y = f(x) = ku(x)$, then

$$f'(x) = ku'(x)$$

$$(y' = ku', \quad \frac{dy}{dx} = k \frac{du}{dx})$$

The derivative of a constant \times a differentiable function is the constant \times the derivative of the function.



3. Limits and the derivative

3-5 Basic Differentiation Properties

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The derivative of a constant \times a differentiable function is the constant \times the derivative of the function.

THEOREM 3 Sum and Difference Property

If $y = f(x) = u(x) \pm v(x)$, then

$$f'(x) = u'(x) \pm v'(x)$$

The derivative of the \pm of two differentiable functions is the \pm of the derivatives of the functions.

Remark: The Sum and Constant Multiple Rule

$$\frac{d}{dx} (af(x) + bg(x)) = a \frac{d}{dx} f(x) + b \frac{d}{dx} g(x) = af'(x) + bg'(x)$$

3. Limits and the derivative

3-5 Basic Differentiation Properties

Example 8:

$f(x) = -3x^2 + 5x - 7$, find $f'(x)$



3. Limits and the derivative

3-5 Basic Differentiation Properties

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$f(x) = -3x^2 + 5x - 7$, find $f'(x)$

$$f'(x) = \frac{d}{dx} (-3x^2 + 5x - 7(1)) \quad \text{Identify multiplicative constants}$$



3. Limits and the derivative

3-5 Basic Differentiation Properties

Example 8:

$f(x) = -3x^2 + 5x - 7$, find $f'(x)$

$$\begin{aligned} f'(x) &= \frac{d}{dx} (-3x^2 + 5x - 7(1)) && \text{Identify multiplicative constants} \\ &= -3 \frac{d}{dx} x^2 + 5 \frac{d}{dx} x - 7 \frac{d}{dx} 1 && \text{Sum and Constant Multiple Rule} \end{aligned}$$



3. Limits and the derivative

3-5 Basic Differentiation Properties

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3. Limits and the derivative

3-5 Basic Differentiation Properties

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$$\begin{aligned}f'(x) &= \frac{d}{dx} (-3x^2 + 5x - 7(1)) && \text{Identify multiplicative constants} \\&= -3 \frac{d}{dx} x^2 + 5 \frac{d}{dx} x - 7 \frac{d}{dx} 1 && \text{Sum and Constant Multiple Rule} \\&= -3(2x^{2-1}) + 5(1) - 7(0) && \text{Power Rule} \\&= -6x + 5\end{aligned}$$



3. Limits and the derivative

3-5 Basic Differentiation Properties

Example 9:

$$f(x) = 3 - \frac{5}{x}, \text{ find } f'(x)$$



3. Limits and the derivative

3-5 Basic Differentiation Properties

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Rewrite f as constants · power function

$$f(x) = 3 - \frac{5}{x} = 3 - 5 \left(\frac{1}{x} \right) = 3 - x^{-1}$$



3. Limits and the derivative

3-5 Basic Differentiation Properties

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Rewrite f as constants · power function

$$f(x) = 3 - \frac{5}{x} = 3 - 5 \left(\frac{1}{x} \right) = 3 - x^{-1}$$

Now take the derivative

$$f'(x) = \frac{d}{dx} (3 - x^{-1})$$



3. Limits and the derivative

3-5 Basic Differentiation Properties

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Now take the derivative

$$\begin{aligned} f'(x) &= \frac{d}{dx} (3 - x^{-1}) \\ &= \frac{d}{dx} 3 - 5 \frac{d}{dx} x^{-1} \end{aligned} \quad \text{Sum and Constant Multiple Rule}$$



3. Limits and the derivative

3-5 Basic Differentiation Properties

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Now take the derivative

$$\begin{aligned} f'(x) &= \frac{d}{dx} (3 - x^{-1}) \\ &= \frac{d}{dx} 3 - 5 \frac{d}{dx} x^{-1} \quad \text{Sum and Constant Multiple Rule} \\ &= 0 - 5(-1 \cdot x^{-1-1}) \end{aligned}$$



3. Limits and the derivative

3-5 Basic Differentiation Properties

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3. Limits and the derivative

3-5 Basic Differentiation Properties

Exercise:

a. $f(x) = 3 - 5\sqrt{5}$, find $f'(x)$

b. $f(x) = \frac{23x^3}{19} - \frac{7}{35x^5}$, find $f'(x)$

c. $f(x) = \frac{2\sqrt[7]{x}}{13} - \frac{3}{17x^{\frac{2}{5}}}$, find $f'(x)$



3. Limits and the derivative

3-5 Basic Differentiation Properties

Applications

Remember that the derivative gives the **instantaneous rate of change** of the function with respect to x . That might be:

- Tangent line slope at a point on the curve of the function
- Instantaneous velocity
- Marginal Cost.

If $C(x)$ is the cost function, that is, the total cost of producing x items, then $C'(x)$ approximates the cost of producing one more item at a production level of x items. $C'(x)$ is called the **marginal cost**.



3. Limits and the derivative

3-5 Basic Differentiation Properties

Application Example: Tangent Line Problems

Goal: Given formula for a function $f(x)$, find the line equation for the line that is tangent to graph of $f(x)$ at $x = a$.

Things we know about the tangent line

Graphical interpretation:

- The tangent line touches the graph of f at $x = a$
- The tangent line "looks like it is going the same direction as the graph of f at that point"

Analytical definition:

- The tangent line passes through the point $(x, y) = (a, f(a))$
- The tangent line has slope $m = f'(a)$



3. Limits and the derivative

3-5 Basic Differentiation Properties

Recall: "point slope" form of the equation of a line

If we know that a line

- passes through know point $(x, y) = (a, b)$
- has slope m

Then the point slope form of the equation for the line is:

$$y - b = m(x - a)$$

Apply this to the tangent line

- passes through the point $(x, y) = (a, f(a))$
- has slope $m = f'(a)$

Build the equation for the line

$$y - f(a) = f'(a)(x - a) \quad \text{equation of the tangent line}$$



3. Limits and the derivative

3-5 Basic Differentiation Properties

Example 10:

$f(x) = x^3 - 9x^2 + 15x + 25$, find the equation of the line that is tangent to graph of f at $x = 2$.

3. Limits and the derivative

3-5 Basic Differentiation Properties

Example 10:

$f(x) = x^3 - 9x^2 + 15x + 25$, find the equation of the line that is tangent to graph of f at $x = 2$.

We need to build this equation

$$y - f(a) = f'(a)(x - a)$$

$a = 2$ (x -coord of point of tangency)

$f(2) = 2^3 - 9 \cdot 2^2 + 15 \cdot 2 + 25 = 8 - 36 + 30 + 25 = 27$ (y -coord of point of tangency)

3. Limits and the derivative

3-5 Basic Differentiation Properties

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$f'(x) = \frac{d}{dx}(x^3 - 9x^2 + 15x + 25) = 3x^2 - 18x + 15$

3. Limits and the derivative

3-5 Basic Differentiation Properties

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$f'(a) = f'(2) = 3(2)^2 - 18(2) + 15 = -9$ (slope of the tangent line)

3. Limits and the derivative

3-5 Basic Differentiation Properties

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$$f'(x) = \frac{d}{dx}(x^3 - 9x^2 + 15x + 25) = 3x^2 - 18x + 15$$

$f'(a) = f'(2) = 3(2)^2 - 18(2) + 15 = -9$ (slope of the tangent line) Substitute the parts into the equation:

$$y - 27 = (-9)(x - 2) \quad \text{point slope form of the equation}$$

Convert to slope intercept form

$$y - 27 = (-9)(x - 2)$$

$$y = -9x + 18 + 27$$

$$y = -9x + 45 \quad \text{slope intercept form of the equation}$$

3. Limits and the derivative

3-5 Basic Differentiation Properties

Application Example: Marginal Cost

The total cost of producing x laptop per day is

$$C(x) = 1000 + 100x - 0.5x^2, \quad \text{for } 0 \leq x \leq 100$$

- 1 Find the marginal cost of production at a production level of x laptops.
- 2 Find the marginal cost of production at a production level of 80 laptops.
- 3 Find the actual cost of producing the 81st laptop and compare this with the marginal cost.

3. Limits and the derivative

3-5 Basic Differentiation Properties

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1 The marginal cost will be: $C'(x) = 100 - x$

3. Limits and the derivative

3-5 Basic Differentiation Properties

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2 $C'(80) = 100 - 80 = 20$ It will cost approximately \$20 to produce the 81st laptop ($C'(x)$ approximates the cost of producing 1 more item at a production level of x items)

3. Limits and the derivative

3-5 Basic Differentiation Properties

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2 $C'(80) = 100 - 80 = 20$ It will cost approximately \$20 to produce the 81st laptop ($C'(x)$ approximates the cost of producing 1 more item at a production level of x items)

3 The actual cost of the 81st laptop will be

$C(81) - C(80) = \$5819.50 - \$5800 = \$19.50$. This is approximately equal to the marginal cost.

3. Limits and the derivative

1 3-1 Introduction to limits

2 3-2 Infinite Limits and Limits at Infinity

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3. Limits and the derivative

3-6 Differentials

Learning Objectives

- Evaluate increments.
- Evaluate differentials.
- Use differentials to approximate increments.



3. Limits and the derivative

3-6 Differentials

Example 1:

Let $y = f(x) = x^2$.



3. Limits and the derivative

3-6 Differentials

Example 1:

Let $y = f(x) = x^2$.

If x changes from 2 to 2.5, then y will change from $y = f(2) = 4$ to $y = f(2.5) = 6.5$.



3. Limits and the derivative

3-6 Differentials

Example 1:

Let $y = f(x) = x^2$.

If x changes from 2 to 2.5, then y will change from $y = f(2) = 4$ to $y = f(2.5) = 6.5$.

We can write this using increment notation.

- The change in x is called the **increment in x** and is denoted by Δx .
- The change in y is called the **increment in y** and is denoted by Δy .

In this example,

$$\Delta x = 2.5 - 2 = 0.5$$

$$\Delta y = f(2.5) - f(2) = 6.5 - 4 = 1.5$$



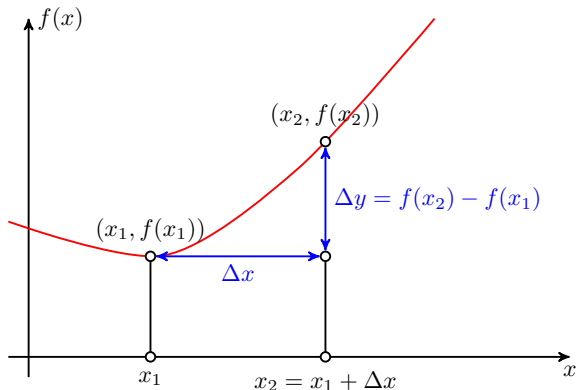
3. Limits and the derivative

3-6 Differentials

DEFINITION Increments

For $y = f(x)$, $\Delta x = x_2 - x_1$, $\Delta y = y_2 - y_1$

- Δy represents the change in y corresponding to a Δx change in x .
- Δx can be either positive or negative.



3. Limits and the derivative

3-6 Differentials

Differentials

Assume that the limit

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}$$

exists.



3. Limits and the derivative

3-6 Differentials

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For small Δx ,

$$f'(x) \approx \frac{\Delta y}{\Delta x}$$



3. Limits and the derivative

3-6 Differentials

Differentials

Assume that the limit

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x}$$

exists.

For small Δx ,

$$f'(x) \approx \frac{\Delta y}{\Delta x}$$

Multiplying both sides of this equation by Δx gives us

$$\Delta x \cdot f'(x) \approx \cancel{\Delta x} \frac{\Delta y}{\cancel{\Delta x}}$$

$$\Delta y \approx f'(x)\Delta x$$

Here the increments Δx and Δy represent the actual changes in x and y .



3. Limits and the derivative

3-6 Differentials

Differentials (continued)

One of the notation for the derivative is: $f'(x) = \frac{dy}{dx}$.

Multiplying both sides of this equation by dx gives us

$$dy = f'(x)dx$$

We treat this equation as a definition, and call dx and dy **differentials**.



3. Limits and the derivative

3-6 Differentials

Differentials (continued)

One of the notation for the derivative is: $f'(x) = \frac{dy}{dx}$.

Multiplying both sides of this equation by dx gives us

$$dy = f'(x)dx$$

We treat this equation as a definition, and call dx and dy **differentials**.

- Δx and dx are the same, and represent the change in x .
- Δy stands for the **actual change** in y resulting from the change in x .
- dy stands for the **approximate change** in y estimated by using derivatives

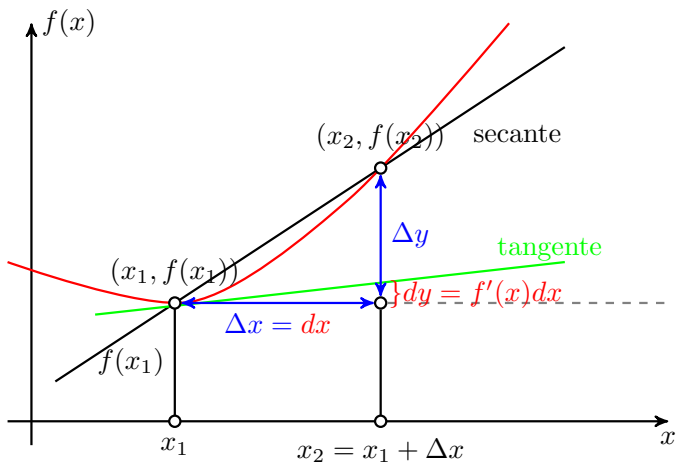
$$\Delta y \approx dy = f'(x)dx$$

In application, we use dy (which is easy to calculate) to estimate Δy (which is what we want).



3. Limits and the derivative

3-6 Differentials



3. Limits and the derivative

3-6 Differentials

Example 2:

Find dy for $f(x) = x^2 + 6x$ and evaluate dy for $x = 2$ and $dx = 0.1$.



3. Limits and the derivative

3-6 Differentials

Example 2:

Find dy for $f(x) = x^2 + 6x$ and evaluate dy for $x = 2$ and $dx = 0.1$.
Using the definition of the differential, we have

$$dy = f'(x)dx = (2x + 6)dx$$

When $x = 2$ and $dx = 0.1$, $dy = (2(2) + 6)\frac{1}{10} = 1$



3. Limits and the derivative

3-6 Differentials

Example 3:

A company manufactures and sells x laptops per week. If the weekly cost and revenue equations are

$$\begin{cases} C(x) = 5000 + 2x \\ R(x) = 10x - \frac{x^2}{1000} \\ 0 \leq x \leq 8000 \end{cases} \quad (1)$$

Find the approximate changes in revenue and profit if production is increased from 1000 to 1010 units/week.



3. Limits and the derivative

3-6 Differentials

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Find the approximate changes in revenue and profit if production is increased from 1000 to 1010 units/week.

The profit is

$$P(x) = R(x) - C(x) = 10x - \frac{x^2}{1000} - 5000 - 2x = 8x - \frac{x^2}{1000} - 5000$$



3. Limits and the derivative

3-6 Differentials

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We'll approx ΔR and ΔP with dR and dP .



3. Limits and the derivative

3-6 Differentials

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Find the approximate changes in revenue and profit if production is increased from 1000 to 1010 units/week.

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$$P(x) = R(x) - C(x) = 10x - \frac{x^2}{1000} - 5000 - 2x = 8x - \frac{x^2}{1000} - 5000$$

We'll approx ΔR and ΔP with dR and dP .

$$dR(x) = R'(x)dx = \frac{d}{dx} \left(10x - \frac{x^2}{1000} \right) dx = \left(10 - \frac{x}{500} \right) dx$$

$$dP(x) = P'(x)dx = \left(8 - \frac{x}{500} \right) dx$$

Here $x = 1000$ and $dx = 10$, $dR = (10 - 2)10 = \$80/\text{week}$ $dP = (8 - 2)10 = \$60/\text{week}$.

3. Limits and the derivative

3-6 Differentials

Exercises:

- Find the indicated quantities for $f(x) = 2x^2$
 - Δy , Δx and $\frac{\Delta y}{\Delta x}$; given $x_1 = 2$ and $x_2 = 6$.
 - $\frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$; given $x_1 = 2$.
- Evaluate dy and Δy for $f(x) = x^2 - 5$, where $x = 5$ and $dx = \Delta x = 0.01$.
- Find Δy and dy for $y = x - x^2$ when $x = 1$.
- The total monthly profit (in dollars) that Mandy's Painted Murals earns when the company is contracted to paint x murals in a month can be modeled by $P(x) = -5x^2 + 500x$, where $0 \leq x \leq 100$. Use dP to approximate the change in profit if production increased from 40 to 50 murals per month. Compare this value with the actual change in profit ΔP .



3. Limits and the derivative

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3. Limits and the derivative

3-7 Marginal Analysis in Business and Economics

Learning Objectives

- Solve applications involving marginal cost/revenue/profit.
- Solve applications involving marginal average cost/revenue/profit.



3. Limits and the derivative

3-7 Marginal Analysis in Business and Economics

Remember that **marginal** refers to an **instantaneous rate of change**, that is, a **derivative**.

DEFINITION Marginal Cost

If x is the number of units of a product **produced** in some time interval, then

$$\text{Total cost} = C(x)$$

$$\text{Marginal cost} = C'(x)$$



3. Limits and the derivative

3-7 Marginal Analysis in Business and Economics

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3. Limits and the derivative

3-7 Marginal Analysis in Business and Economics

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DEFINITION Marginal Revenue

If x is the number of units of a product **sold** in some time interval, then

$$\text{Total revenue} = R(x)$$

$$\text{Marginal revenue} = R'(x)$$

DEFINITION Marginal Profit

If x is the number of units of a product **produced** and **sold** in some time interval, then

$$\text{Total profit} = P(x) = R(x) - C(x)$$

$$\text{Marginal profit} = P'(x) = R'(x) - C'(x)$$

3. Limits and the derivative

3-7 Marginal Analysis in Business and Economics

Marginal Cost & Exact Cost

Assume that $C(x)$ is the total cost of producing x items. Then the **exact cost** of producing the $(x + 1)$ item is

$$C(x + 1) - C(x).$$

The **marginal cost** is an approximation of the exact cost.

$$C'(x) \approx C(x + 1) - C(x).$$

Note that similar statement are true for revenue and profit



3. Limits and the derivative

3-7 Marginal Analysis in Business and Economics

Example:

The total cost of producing x guitar is

$$C(x) = 1000 + 100x - 0.25x^2$$

- 1 Find the exact cost of producing the 51st guitar
- 2 Use the marginal cost to approximate the cost of producing the 51st guitar



3. Limits and the derivative

3-7 Marginal Analysis in Business and Economics

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- 1 Find the exact cost of producing the 51st guitar
 - 2 Use the marginal cost to approximate the cost of producing the 51st guitar
- 1 The exact cost is

$$\begin{aligned}C(x+1) - C(x) &= 1000 + 100(x+1) - 0.25(x+1)^2 - 1000 - 100x + 0.25x^2 \\ &= 100 - 0.5x - 0.25 = 99.75 - 0.5x\end{aligned}$$

$$\text{So } C(51) - C(50) = 99.75 - 0.5 \cdot 50 = \$74.75$$



3. Limits and the derivative

3-7 Marginal Analysis in Business and Economics

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$$\text{So } C(51) - C(50) = 99.75 - 0.5 \cdot 50 = \$74.75$$

- 2 The marginal cost is $C'(x) = 100 - 0.5x$.
So $C'(50) = 100 - 25 = \$75$



3. Limits and the derivative

3-7 Marginal Analysis in Business and Economics

DEFINITION Marginal Average Cost

If x is the number of units of a product **produced** in some time interval, then

$$\text{Average cost per unit} = \bar{C}(x) = \frac{C(x)}{x}$$

$$\text{Marginal average cost} = \bar{C}'(x) = \frac{d}{dx}\bar{C}(x)$$

DEFINITION Marginal Average Revenue

If x is the number of units of a product **sold** in some time interval, then

$$\text{Average revenue per unit} = \bar{R}(x) = \frac{R(x)}{x}$$

$$\text{Marginal average revenue} = \bar{R}'(x) = \frac{d}{dx}\bar{R}(x)$$



3. Limits and the derivative

3-7 Marginal Analysis in Business and Economics

DEFINITION Marginal Average Profit

If x is the number of units of a product **produced** and **sold** in some time interval, then

$$\text{Average profit per unit} = \bar{P}(x) = \frac{P(x)}{x}$$

$$\text{Marginal average profit} = \bar{P}'(x) = \frac{d}{dx} \bar{P}(x)$$

Warning!

To calculate the **marginal average**, you must calculate the **average first** and then the derivative.



3. Limits and the derivative

3-7 Marginal Analysis in Business and Economics

Exercise:

The total cost of printing x dictionaries is

$$C(x) = 20000 + 10x$$

- 1 Find the average cost per unit if 1000 dictionaries are produced.
- 2 Find the marginal average cost at a production level of 1000 dictionaries.
- 3 Use the results from above to estimate the average cost per dictionary if 1001 dictionaries are produced.



3. Limits and the derivative

3-7 Marginal Analysis in Business and Economics

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- 2 Find the marginal average cost at a production level of 1000 dictionaries.
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1

$$\begin{aligned}\bar{C}(x) &= \frac{C(x)}{x} = \frac{20000 + 10x}{x} \\ \bar{C}(1000) &= \frac{20000 + 10000}{1000} = \$30\end{aligned}$$



3. Limits and the derivative

3-7 Marginal Analysis in Business and Economics

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3. Limits and the derivative

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- 3 Use the results from above to estimate the average cost per dictionary if 1001 dictionaries are produced.

2 The marginal average cost is

$$\begin{aligned}\overline{C}'(x) &= \frac{d}{dx} \left(\frac{C(x)}{x} \right) = \frac{d}{dx} \left(\frac{20000 + 10x}{x} \right) = \frac{-2000}{x^2} \\ \overline{C}'(1000) &= \frac{-20000}{1000^2} = -0.02\end{aligned}$$

This means that if you raise production from 1000 to 1001 dictionaries, the price per book will fall approximately 2 cents.

3. Limits and the derivative

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3. Limits and the derivative

3-7 Marginal Analysis in Business and Economics

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- 1 Find the average cost per unit if 1000 dictionaries are produced.
 - 2 Find the marginal average cost at a production level of 1000 dictionaries.
 - 3 Use the results from above to estimate the average cost per dictionary if 1001 dictionaries are produced.
- 3 Average cost for 1000 dictionaries = \$30
Marginal average cost = -0.02
The average cost per dictionary for 1001 dictionaries would be the average for 1000, plus the marginal average cost, or

$$\$30 + \$(-0.02) = \$29.98$$



3. Limits and the derivative

3-7 Marginal Analysis in Business and Economics

Exercise:

The price-demand equation and the cost function for production of television sets are given by

$$p(x) = 300 - \frac{x}{30}, \quad \text{and} \quad C(x) = 150000 + 30x$$

where x is the number of sets that can be sold at a price $\$p$ per set, and $C(x)$ is the total cost of producing x sets.

- 1 Find marginal cost.
- 2 Find the revenue function in terms of x .
- 3 Find the marginal revenue.
- 4 Find $R'(1500)$.
- 5 Find the profit function in terms of x .
- 6 Find the marginal profit.
- 7 Find $P'(1500)$.

