MATH 209

Fundamental Mathematics II
Winter 2015

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6. Integration

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Learning Objectives

- Use indefinite integral formulas and properties to find antiderivatives.
- Solve applications that require finding antiderivatives.

DEFINITION: **Antiderivative**

Words: F is an antiderivative of f.

Meaning: f is the derivative of F

$$f = F'$$
$$F' = f$$

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$$= \frac{1}{3}\frac{d}{dx}x^3 = \frac{1}{3}3x^2 = x^2 = f(x)$$

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Example 2:

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$$= \frac{1}{3}\frac{d}{dx}x^3 + \frac{d}{dx}17 = \frac{1}{3}3x^2 + 0 = x^2 = f(x)$$

So there is more than one antiderivative of $f(x)$. That is, given any antiderivative $F(x)$ for $f(x)$, then we can make lots of other antiderivatives. Any function of the form $F(x) + c$ will also be an antiderivative of $f(x)$.

THEOREM 1: Antiderivatives

These are the only antiderivates of $f(x)$. That is, if one antiderivative of $f(x)$ is $F(x)$, then all the other antiderivatives of $f(x)$ are of the form

$$F(x) + c$$

(where c is a constant that can be any real number)

Indefinite Integral

Let $f(x)$ be a function. The family of all functions that are antiderivatives of $f(x)$ is called the indefinite integral and has the symbol

$$\int f(x)dx$$

The symbol $\int$ is called an integral sign, and the function $f(x)$ is called integrand. The symbol dx indicates that antidifferentiation is performed whit respect to the variable x.

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The symbol $\int$ is called an integral sign, and the function $f(x)$ is called integrand. The symbol dx indicates that antidifferentiation is performed whit respect to the variable x. By the previous theorem, if $F(x)$ is any antiderivative of $f(x)$, then

$$\int f(x)dx = F(x) + C$$

The arbitrary constant C is called the constant of integration.

Example 3:

Find the indefinite integral of $f(x) = x^2$.

Example 3:

Find the indefinite integral of $f(x) = x^2$. Check:

$$\int f(x)dx = \int x^2 dx = \frac{x^3}{3} + C,$$

because

$$\frac{d}{dx}\left(\frac{x^3}{3} + C\right) = x^2$$

Formulas and Properties

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C, \qquad \text{valid when } n \neq -1$$

$$\int e^x dx = e^x + C$$

$$\int \frac{1}{x} dx = \ln|x| + C$$

$$\int kf(x)dx = k\int f(x)dx$$

$$\int [f(x) \pm g(x)]dx = \int f(x)dx + \int g(x)dx$$

Formulas and Properties

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C, \qquad \text{valid when } n \neq -1$$

Check by finding the derivative

$$\begin{aligned}
\frac{d}{dx}\left(\frac{x^{n+1}}{n+1} + C\right) &= \frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) + \frac{d}{dx}C \\
&= \frac{1}{n+1}\frac{d}{dx}x^{n+1} + 0 \\
&= \frac{1}{n+1}(n+1)x^{(n+1)-1} \\
&= x^n
\end{aligned}$$

Example 4:

$$\int x^8 dx = \frac{x^{8+1}}{8+1} + C = \frac{x^9}{9} + C$$

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Check:

$$\frac{d}{dx}\left(\frac{x^9}{9} + C\right) = \frac{1}{9}9x^8 + 0 = x^8$$

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Check:

$$\frac{d}{dx}\left(\frac{x^9}{9} + C\right) = \frac{1}{9}9x^8 + 0 = x^8$$

Example 5:

$$\int x^{-4} dx = \frac{x^{-4+1}}{-4+1} + C = \frac{x^{-3}}{-3} + C$$

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Check:

$$\frac{d}{dx}\left(\frac{x^9}{9} + C\right) = \frac{1}{9}9x^8 + 0 = x^8$$

Example 5:

$$\int x^{-4} dx = \frac{x^{-4+1}}{-4+1} + C = \frac{x^{-3}}{-3} + C$$

Check:

$$\frac{d}{dx}\left(\frac{x^{-3}}{-3} + C\right) = -3\frac{x^{-3-1}}{-3}x^{-3-1} = x^{-4}$$

Example 6:

$$\int x^{2/3} dx = \frac{x^{\frac{2}{3}+1}}{\frac{2}{3}+1} + C = \frac{x^{\frac{10}{3}}}{\frac{10}{3}} + C = \frac{3}{10} x^{\frac{10}{3}} + C$$

Example 6:

$$\int x^{2/3}dx = \frac{x^{\frac{2}{3}+1}}{\frac{2}{3}+1} + C = \frac{x^{\frac{10}{3}}}{\frac{10}{3}} + C = \frac{3}{10}x^{\frac{10}{3}} + C$$

Example 7:

$$\int x dx = \frac{x^{1+1}}{1+1} + C = \frac{x^2}{2} + C$$

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Example 7:

$$\int xdx = \frac{x^{1+1}}{1+1} + C = \frac{x^2}{2} + C$$

Example 8:

$$\int 1dx = \int x^0dx = \frac{x^{0+1}}{0+1} + C = x + C$$

Application:

In spite of the prediction of a paperless computerized office, paper and paperboard production in the United States has steadily increased. In 1990 the production was 80.3 million short tons, and since 1970 production has been growing at a rate given by

$$f'(x) = 0.048x + 0.95,$$

where x is years after 1970.
Find $f(x)$ and the production levels in 1970 and 2000.

Application:

We need the integral of $f'(x)$

$$f(x) = \int f'(x)dx = \int (0.048x + 0.95)dx = 0.048\frac{x^2}{2} + 0.95x + C = 0.24x^2 + 0.95x + C$$

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We need the integral of $f'(x)$

$$f(x) = \int f'(x)dx = \int (0.048x + 0.95)dx = 0.048\frac{x^2}{2} + 0.95x + C = 0.24x^2 + 0.95x + C$$

Noting that $f(20) = 80.3$, we calculate

$$\begin{aligned} 80.3 &= 0.24(20)^2 + 0.95 \cdot 20 + C \\ 80.3 &= 28.6 + C \\ C &= 51.7 \end{aligned}$$

So $f(x) = 0.24x^2 + 0.95x + 51.7$

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So $f(x) = 0.24x^2 + 0.95x + 51.7$
The years 1970 and 2000 correspond to $x = 0$ and $x = 30$.

$$\begin{aligned} f(0) &= 51.7 \\ f(30) &= 0.24(30)^2 + 0.95 \cdot 30 + 51.7 \\ &= 101.8 \end{aligned}$$

The production was 51.7 short tons in 1970, and 101.8 short tons in 2000.

Exercises:

1. Find each indefinite integral.
$a.\ \int 5dx \quad b.\ \int edx \quad c.\ \int 4x^3dx \quad d.\ \int tdt$
$e.\ \int u^{21}du \quad f.\ \int -7\sqrt{x}dx \quad g.\ \int \frac{2}{w}dw \quad h.\ \int -e^x$

2. Find the antiderivatives for each derivative.
$a.\ \frac{dy}{dx} = 12 - x^5 \quad b.\ \frac{dy}{dt} = 1 - 2t + t^3 \quad c.\ \frac{du}{dv} = \frac{2}{v} + \frac{v}{2}$

3. Find each indefinite integral.
$a.\ \int 3x(x^2-1)dx \quad \int \left(\frac{x^2}{3} - \frac{3}{x^2} + \frac{3}{x}\right)dx \quad c.\ \int \frac{t^2-t}{t}dt$

4. Find the antiderivative of $\frac{dy}{dx} = e^x - 1$ if $y(0) = 5$.

5. The marginal profit from the sales of x items is given by $P'(x) = -0.01x + 450$. Find $P(x)$ if $P(100) = 2500$.

6. Integration

Learning Objectives

- Integrate expressions by reversing the chain rule.
- Integrate expressions using substitution.

Reversing the Chain Rule

Recall the chain rule:

$$\frac{d}{dx}f[g(x)] = f'[g(x)]g'(x)$$

Reading it backwards, this implies that

$$\int f'[g(x)]g'(x)dx = \int \frac{d}{dx}f[g(x)]dx = f[g(x)] + C$$

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$$\int f'[g(x)]g'(x)dx = \int \frac{d}{dx} f[g(x)]dx = f[g(x)] + C$$

Special Cases:

$$\int [f(x)]^n f'(x)dx = \frac{[f(x)]^{n+1}}{n+1} + C, \quad n \neq -1$$

$$\int e^{f(x)} f'(x)dx = e^{f(x)} + C$$

$$\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + C$$

Example:

$$\int (x^5 - 2)^3 (5x^4) dx$$

Note that the derivative of $x^5 - 2 = 5x^4$, the integral appears to be in the chain rule form

$$\int [f(x)]^n f'(x) dx = \frac{[f(x)]^{n+1}}{n+1} + C, \quad n \neq -1$$

with $f(x) = x^5 - 2$ and $n = 3$

Example:

$$\int (x^5-2)^3(5x^4)dx$$

Note that the derivative of $x^5-2=5x^4$, the integral appears to be in the chain rule form

$$\int [f(x)]^n f'(x)dx = \frac{[f(x)]^{n+1}}{n+1}+C, \quad n \neq -1$$

with $f(x)=x^5-2$ and $n=3$
It follows that

$$\int (x^5-2)^3(5x^4)dx = \frac{(x^5-2)^4}{4}+C$$

DEFINITION **Differentials**

If $y = f(x)$ is a differentiable function, then

- The differential dx of the independent variable x is any arbitrary real number.
- The differential dy of the dependent variable y is defined as

$$dy = f'(x)dx$$

Examples:

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- if $y = f(x) = x^5 - 2$, then
$$dy = f'(x)dx = 5x^4dx$$
- if $y = f(x) = e^{5x}$, then
$$dy = f'(x)dx = 5e^{5x}dx$$
- if $y = f(x) = \ln(3x - 5)$, then

Examples:

- if $y = f(x) = x^5 - 2$, then

$$dy = f'(x)dx = 5x^4 dx$$

- if $y = f(x) = e^{5x}$, then

$$dy = f'(x)dx = 5e^{5x} dx$$

- if $y = f(x) = \ln(3x - 5)$, then

$$dy = f'(x)dx = \frac{3}{3x-5} dx$$

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$$dx = \frac{du}{2x}$$

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Find $\int (x^2+1)^5 2x dx$ For our substitution, let $u = x^2+1$, then $\frac{du}{dx} = 2x$, use it to build dx

$$dx = \frac{du}{2x}$$

The integral becomes

$$\int (x^2+1)^5 2x dx = \int u^5 2x \frac{du}{2x} = \int u^5 du = \frac{u^6}{6} + C$$

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The integral becomes

$$\int (x^2+1)^5 2x dx = \int u^5 2x \frac{du}{2x} = \int u^5 du = \frac{u^6}{6} + C$$

Substitute $u = x^2+1$

$$\int (x^2+1)^5 2x dx = \frac{(x^2+1)^6}{6} + C$$

General Indefinite Integral Formulas

$$\int u^n du = \frac{u^{n+1}}{n+1} + C, \quad n \neq -1$$

$$\int e^u du = e^u + C$$

$$\int \frac{1}{u} du = \ln|u| + C$$

PROCEDURE Integration by Substitution

- Step 1 Select a substitution that appears to simplify the integrand. In particular, try to select u so that du is a factor of the integrand.
- Step 2 Express the integrand entirely in terms of u and du, completely eliminating the original variable.
- Step 3 Evaluate the new integral, if possible.
- Step 4 Express the antiderivative found in step 3 in terms of the original variable.

Example 2:

Find $\int (x^3 - 5)^4 3x^2 dx$

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Step 1 Select u.
Let $u = x^3 - 5$, then $du = 3x^2 dx$

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Find $\int (x^3 - 5)^4 3x^2 dx$

Step 1 Select u.
Let $u = x^3 - 5$, then $du = 3x^2 dx$

Step 2 Express integral in terms of u.

$$\int (x^3 - 5)^4 3x^2 dx = \int u^4 du$$

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Step 2 Express integral in terms of u.

$$\int (x^3 - 5)^4 3x^2 dx = \int u^4 du$$

Step 3 Integrate.

$$\int u^4 du = \frac{u^5}{5} + C$$

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Step 2 Express integral in terms of u.

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Step 3 Integrate.

$$\int u^4 du = \frac{u^5}{5} + C$$

Step 4 Express the answer in terms of x.

$$\int (x^3 - 5)^4 3x^2 dx = \frac{(x^3 - 5)^5}{5} + C$$

Example 3:

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Step 1 Select u.
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$$\int (x^2+5)^{\frac{1}{2}} 2xdx = \int u^{\frac{1}{2}} du$$

Step 3 Integrate.

$$\int u^{\frac{1}{2}} du = \frac{2}{3} u^{\frac{3}{2}} + C$$

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Step 1 Select u.
Let $u = x^2 + 5$, then $du = 2xdx$

Step 2 Express integral in terms of u.

$$\int (x^2+5)^{\frac{1}{2}} 2x dx = \int u^{\frac{1}{2}} du$$

Step 3 Integrate.

$$\int u^{\frac{1}{2}} du = \frac{2}{3} u^{\frac{3}{2}} + C$$

Step 4 Express the answer in terms of x.

$$\int (x^3-5)^4 3x^2 dx = \frac{2}{3}(x^2+5)^{\frac{3}{2}} + C$$

Example 4:

Find $\int (x^3 - 5)^4 x^2 dx$

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Find $\int (x^3 - 5)^4 x^2 dx$

Step 1 Select u.
Let $u = x^3 - 5$, then $du = 3x^2 dx$, or $dx = \frac{1}{3x^2} du$

Example 4:

Find $\int (x^3-5)^4 x^2 dx$

Step 1 Select u.
Let $u = x^3 - 5$, then $du = 3x^2 dx$, or $dx = \frac{1}{3x^2} du$

Step 2 Express integral in terms of u.

$$\int (x^3-5)^4 x^2 dx = \int u^4 \frac{x^2}{3x^2} du = \frac{1}{3} \int u^4 du$$

Example 4:

Find $\int (x^3-5)^4 x^2 dx$

Step 1 Select u.
Let $u = x^3 - 5$, then $du = 3x^2 dx$, or $dx = \frac{1}{3x^2} du$

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$$\int (x^3-5)^4 x^2 dx = \int u^4 \frac{x^2}{3x^2} du = \frac{1}{3} \int u^4 du$$

Step 3 Integrate.

$$\frac{1}{3} \int u^4 du = \frac{1}{3} \frac{u^5}{5} + C$$

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Let $u = x^3 - 5$, then $du = 3x^2 dx$, or $dx = \frac{1}{3x^2} du$

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$$\int (x^3-5)^4 x^2 dx = \int u^4 \frac{x^2}{3x^2} du = \frac{1}{3} \int u^4 du$$

Step 3 Integrate.

$$\frac{1}{3} \int u^4 du = \frac{1}{3} \frac{u^5}{5} + C$$

Step 4 Express the answer in terms of x.

$$\int (x^3-5)^4 x^2 dx = \frac{1}{15}(x^3-5)^5 + C$$

Example 5:

Find $\int x^2 e^{4x^3} dx$

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Step 1 Select u.
Let $u = 4x^3$, then $du = 12x^2 dx$, or $dx = \frac{1}{12x^2} du$

Example 5:

Find $\int x^2 e^{4x^3} dx$

Step 1 Select u.
Let $u = 4x^3$, then $du = 12x^2 dx$, or $dx = \frac{1}{12x^2} du$

Step 2 Express integral in terms of u.

$$\int x^2 e^{4x^3} dx = \int e^u \frac{x^2}{12x^2} du = \frac{1}{12} \int e^u du$$

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Find $\int x^2 e^{4x^3}\,dx$

Step 1 Select u.
Let $u = 4x^3$, then $du = 12x^2 dx$, or $dx = \frac{1}{12x^2} du$

Step 2 Express integral in terms of u.

$$\int x^2 e^{4x^3}\,dx = \int e^u \frac{x^2}{12x^2}\,du = \frac{1}{12}\int e^u\,du$$

Step 3 Integrate.

$$\frac{1}{12}\int e^u\,du = \frac{1}{12}e^u + C$$

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Step 1 Select u.
Let $u = 4x^3$, then $du = 12x^2 dx$, or $dx = \frac{1}{12x^2} du$

Step 2 Express integral in terms of u.

$$\int x^2 e^{4x^3} dx = \int e^u \frac{x^2}{12x^2} du = \frac{1}{12} \int e^u du$$

Step 3 Integrate.

$$\frac{1}{12} \int e^u du = \frac{1}{12} e^u + C$$

Step 4 Express the answer in terms of x.

$$\int x^2 e^{4x^3} dx = \frac{1}{12} e^{4x^3} + C$$

Example 6:

Find $\int \frac{x}{(5-2x^2)^5} dx$

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Find $\int \frac{x}{(5-2x^2)^5}dx$

Step 1 Select u.

Let $u = 5 - 2x^2$, then $du = -4xdx$, or $dx = -\frac{1}{4x}du$

Example 6:

Find $\int \frac{x}{(5-2x^2)^5} dx$

Step 1 Select u.
Let $u = 5 - 2x^2$, then $du = -4x dx$, or $dx = -\frac{1}{4x} du$

Step 2 Express integral in terms of u.

$$\int \frac{x}{(5-2x^2)^5} dx = \int \frac{x}{u^5} \frac{-1}{4x} du = -\frac{1}{4} \int \frac{1}{u^5} du$$

Example 6:

Find $\int \frac{x}{(5-2x^2)^5} dx$

Step 1 Select u.

Let $u = 5 - 2x^2$, then $du = -4x dx$, or $dx = -\frac{1}{4x} du$

Step 2 Express integral in terms of u.

$$\int \frac{x}{(5-2x^2)^5} dx = \int \frac{x}{u^5} \frac{-1}{4x} du = -\frac{1}{4} \int \frac{1}{u^5} du$$

Step 3 Integrate.

$$-\frac{1}{4} \int u^{-5} du = (-\frac{1}{4})(\frac{-1}{4}) u^{-4} + C$$

Example 6:

Find $\int \frac{x}{(5-2x^2)^5} dx$

Step 1 Select u.
Let $u = 5 - 2x^2$, then $du = -4x dx$, or $dx = -\frac{1}{4x} du$

Step 2 Express integral in terms of u.

$$\int \frac{x}{(5-2x^2)^5} dx = \int \frac{x}{u^5} \frac{-1}{4x} du = -\frac{1}{4} \int \frac{1}{u^5} du$$

Step 3 Integrate.

$$-\frac{1}{4} \int u^{-5} du = (-\frac{1}{4})(\frac{-1}{4}) u^{-4} + C$$

Step 4 Express the answer in terms of x.

$$\int \frac{x}{(5-2x^2)^5} dx = \frac{1}{16(5-2x^2)^4} + C$$

Example 7:

Find $\int x(x+6)^8 dx$

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Step 1 Let $u = x + 6$, then $du = dx$.

Example 7:

Find $\int x(x+6)^8 dx$

Step 1 Let $u = x + 6$, then $du = dx$.

Step 2 Express integral in terms of u.

$$\int x(x+6)^8 dx = \int x u^8 du$$

We need to get rid of the x, and express it in term of u: $x = u - 6$.

$$\int x u^8 du = \int (u-6) u^8 du = \int (u^9 - 6u^8) du$$

Example 7:

Find $\int x(x+6)^8 dx$

Step 1 Let $u = x + 6$, then $du = dx$.

Step 2 Express integral in terms of u.

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Step 4

$$\int x(x+6)^8 dx = \frac{(x+6)^{10}}{10} - 2\frac{(x+6)^9}{3} + C$$

Application:

The marginal price of a supply level of x bottles of baby shampoo per week is given by

$$p'(x) = \frac{300}{(3x+25)^2}$$

Find the price-supply equation if the distributor of the shampoo is willing to supply 75 bottles a week at a price of $\$1.60$ per bottle.

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$$p(x) = \int p'(x)dx = \int \frac{300}{(3x+25)^2}dx$$

Let $u = 3x + 25$, then $du = 3dx$ or $dx = \frac{1}{3}du$.

$$\begin{aligned} p(x) &= \int p'(x)dx = \int \frac{300}{(3x+25)^2}dx \\ &= \int \frac{300}{u^2}\frac{1}{3}du = 100\int \frac{1}{u^2}du = \frac{-100}{u} + C \\ &= \frac{-100}{3x+25} + C \end{aligned}$$

Application:

Now we need to find C using the fact that 75 bottles sell for $\$1.60$ per bottle.

$$p(x) = \frac{-100}{3x+25} + C$$
$$p(75) = 1.60 = \frac{-100}{3(75)+25} + C = \frac{-100}{250} + C = -0.4 + C$$
$$C = 2$$

So $p(x) = \frac{-100}{3x+25} + 2$

Exercises:

1. Find each indefinite integral and check the result by differentiating.

$$a.\ \int 6x^5(x^6+1)^3dx \quad b.\ \int \frac{x^2}{(5-x^3)^4}dx \quad c.\ \int 10x(x^2-1)^5dx$$
$$d.\ \int 7e^{7x}dx \quad e.\ \int (x-1)e^{x^2-2x}dx \quad f.\ \int \frac{x}{x^2+2}dx$$
$$g.\ \int \frac{1}{(3t+4)^2}dt \quad h.\ \int x\sqrt{x+1}dx \quad i.\ \int \frac{x}{\sqrt{x-7}}dx$$

2. The weekly marginal revenue from the sale of x designer leather belts is given by

$$R'(x) = 30 - 0.001x + \frac{300}{x}, \qquad R(1) = 50,$$

where $R(x)$ is revenue in dollars.
Find the revenue function. Find the revenue from the sale of 500 leather belts.

6. Integration

Learning Objectives

- Solve basic differential equations.
- Solve applications involving differential equations.

Intro to Differential Equations

We previously studied equations like

$$\frac{dy}{dx} = 3x^2 - 5x + 3$$
$$p'(x) = 300e^{-0.05x}$$
$$y'' = 3x^2 - 5x + 4$$

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$$y'' = 3x^2 - 5x + 4$$

These are examples of differential equations.
The first two are first order differential equations involving only the first derivative while the last is a second order differential equation because it involves the second derivative.

Differential Equations and Slope Fields

A slope field is a graphical representation of the solutions of a differential equation. It is useful because it can be created without solving the differential equation analytically. The representation may be used to qualitatively visualize solutions, or to numerically approximate them.

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Slope fields will be introduced through an example.
Let

$$\frac{dy}{dx} = y + 1$$

Remember the geometric interpretation of the derivative is the slope of the line at the given point.

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For the given function, the slope at the point $(x, y) = (0, 2)$ would be 3. At the points $(-2, 1)$ and $(2, 3)$, the slopes would be 2 and 4 respectively.

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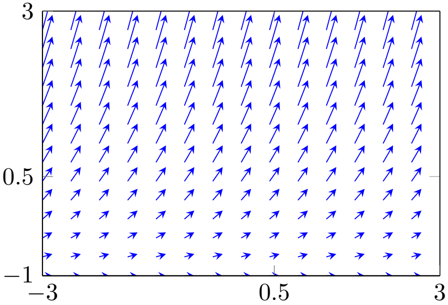
Let

$$\frac{dy}{dx} = y + 1$$

Remember the geometric interpretation of the derivative is the slope of the line at the given point.
For the given function, the slope at the point $(x, y) = (0, 2)$ would be 3. At the points $(-2, 1)$ and $(2, 3)$, the slopes would be 2 and 4 respectively.
A slope field places a short line segment at each point on the graph indicating the slope of the function at that point.

Differential Equations and Slope Fields

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y + 1$$



The graph of the slope field for all points.

Differential Equations and Slope Fields

We claim that the solution of

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is

$$y = Ce^x - 1.$$

Let's check to see if it works. Substitute in the original equation for y.

$$\begin{aligned} \frac{d}{dx}\left(Ce^x - 1\right) &= Ce^x \\ &= (Ce^x - 1) + 1 \\ &= y + 1 \end{aligned}$$

We have confirmed that $y = Ce^x - 1$ is a solution to the differential equation $\frac{dy}{dx} = y + 1$.

Differential Equations and Slope Fields

$y = Ce^x - 1$ is the general solution to the differential equation $\frac{dy}{dx} = y + 1$.
A particular solution going through the point $(0, 0)$ would be $0 = Ce^0 - 1$ or $C = 1$. This yields the equation

$$y = e^x - 1$$

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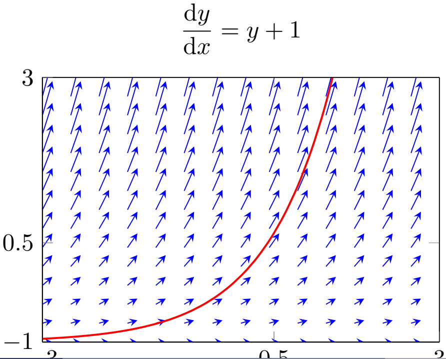
We will now graph that equation on our slope field:



Slope Fields in General

A slope field for a first-order differential equation is obtained by drawing tangent line segments determined by the equation at each point in a grid.

Slope Fields in General

A slope field for a first-order differential equation is obtained by drawing tangent line segments determined by the equation at each point in a grid.
In general, this is done by computers, but to obtain a few points by hand we can do the following:

- Draw tangent lines for a solution curve of the differential equation that passes through a few points.
- Sketch an approximate graph of the solution curve that passes through these points.
- Of all the elementary functions previously discussed, make a conjecture as to what type of function appears to be a solution to the differential equation.

Continuous Compound Interest Revisited

Let P be the initial amount of money deposited in an account ($A(0) = P$).
Let A be the amount at any time t.

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Since $\frac{dA}{dt}$ is the rate of growth of A with respect to t, we have

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where r is an appropriate constant.

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$$\frac{dA}{dt} = rA$$

where r is an appropriate constant.
To find the function $A = A(t)$ that satisfies the above conditions we divide both sides of the above equation by A and integrate with respect to time t.

$$\frac{1}{A}\frac{dA}{dt} = r$$

Continuous Compound Interest Revisited

Now we integrate each side with respect to t:

$$\int \frac{1}{A}\frac{dA}{\cancel{dt}}\cancel{dt} = \int r dt$$

$$\int \frac{1}{A} dA = \int r dt$$

$$\ln |A| = rt + C$$

$$\ln A = rt + C$$

$$A = e^{rt+C} = e^{C} e^{rt}$$

Continuous Compound Interest Revisited

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$$\int \frac{1}{A}\frac{dA}{\cancel{dt}}\cancel{dt} = \int r dt$$
$$\int \frac{1}{A} dA = \int r dt$$
$$\ln|A| = rt + C$$
$$\ln A = rt + C$$
$$A = e^{rt+C} = e^C e^{rt}$$

Since $A(0) = P$, we evaluate $A(t) = e^C e^{rt}$ at $t = 0$ and set the result equal to P:

$$A(0) = e^C e^0 = e^C = P$$

Hence, $e^C = P$ and we can rewrite $A(t) = e^C e^{rt}$ in the form $A = Pe^{rt}$. This is the same continuous compound interest formula obtained in Section 5-1, where the principal P is invested at an annual nominal rate r compounded continuously for t years.

Exponential Growth Law

In general, the rate of growth of money in the previous case may be extended to any quantity Q that grows proportionally to the amount present with respect to time. We obtain the following theorem:

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THEOREM 1 **Exponential Growth Law**

If $\frac{dQ}{dt} = rQ$ and $Q(0) = Q_0$, then $Q = Q_0 e^{rt}$, where

- $Q_0 =$ amount of money at $t = 0$
- $r =$ relative growth rate (expressed as a decimal)
- $t =$ time
- $Q =$ quantity at time t

If $r > 0$, then $\frac{dQ}{dt} > 0$ and Q is increasing, this becomes exponential growth.
If $r < 0$, then $\frac{dQ}{dt} < 0$ and Q is decreasing this becomes an exponential decay problem.

Relative Growth Rate

The constant r in the exponential growth law is called the relative growth rate.
If the relative growth rate is $r = 0.02$, then the quantity Q is growing at a rate $\frac{dQ}{dt} = 0.02Q$ (that is 2% of the quantity Q per unit of time t).

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Note the distinction between the relative growth rate r and the rate of growth $\frac{dQ}{dt}$ of the quantity Q.
Relative growth rate is 0.02 and the rate of growth is $0.02Q$.
Once we know that the rate of growth of something is proportional to the amount present, we know that it has exponential growth and we can use the exponential growth formula.

Example 1:

China had a population of 1.32 billion in 2007 ($t = 0$). Let P represent the population (in billions) t years after 2007, and assume a continuous growth rate of 0.6%. Find the estimated population for China in the year 2025.

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Find the estimated population for China in the year 2025.
The exponential growth/decay law applies, so that

$$P = 1.32e^{0.006t}$$

Substituting $18 = 25 - 7$ for t yields

$$P = 1.32e^{0.006 \cdot 18} = 1.47$$

billion people

Example 2:

A bone from an ancient tomb was discovered and was found to have 5% of the original radioactive carbon present. The continuous compound rate of decay for radioactive carbon-14 is 0.0001238.
Estimate the age of the bone.

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A bone from an ancient tomb was discovered and was found to have 5% of the original radioactive carbon present. The continuous compound rate of decay for radioactive carbon-14 is 0.0001238.
Estimate the age of the bone.
The exponential growth/decay law that applies is

$$\frac{dQ}{dt} = -0.0001238Q, \quad Q(0) = Q_0$$

The solution is

$$Q(t) = Q_0 e^{-0.0001238t} = 0.05Q_0$$

So

$$\begin{aligned} 0.05 &= e^{-0.0001238t} \\ \ln 0.05 &= \ln\left(e^{-0.0001238t}\right) \\ t &= \frac{\ln 0.05}{-0.0001238} \approx 24,198 \text{ years} \end{aligned}$$

Exercises:

1. Find the general or particular solution, as indicated, for each differential equation.

$$a.\ \frac{dy}{dx} = e^{0.035x} \qquad b.\ \frac{dy}{dx} = \sqrt[3]{x} \qquad \frac{dy}{dx} = \frac{1}{3x-5}; y(2) = 3$$

2. Find the general or particular solution, as indicated, for each differential equation.

$$a.\ \frac{dy}{dx} = 4y \qquad b.\ \frac{dx}{dt} = 0.3x; x(0) = 50$$

3. Show that $y = Ce^{-x} + 3$ is a solution to the differential equation $\frac{dy}{dx} = 3 - y$ for any real number C. Find the particular solution that passes through the point $(0, 0)$.

4. Find the amount A in an account after t years if $\frac{dA}{dt} = 0.065A$ and $A(0) = 10,000$.

5. The marginal price $\frac{dp}{dx}$ at x units of supply per day is proportional to the price p. There is no weekly demand at a price of $\$300$ per unit ($p(0) = 300$), and there is a weekly demand of 10 units at a price $\$250$ per unit ($p(10) = 250$).

- Find the price-demand equation.
- At a demand of 20 units per week, what is the price?

6. Integration

Learning Objectives

- Approximate areas using left and right sums.
- Use properties of definite integrals to solve problems.

Introduction

We have been studying the indefinite integral or antiderivative of a function. We will be interested in the area between graph of a function $f(x)$ and the $x-$axis from $x = a$ and $x = b$.

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We now introduce the definite integral. This integral will be the area bounded by $f(x)$, the x axis, and the vertical lines $x = a$ and $x = b$, with notation

$$\int_a^b f(x)dx$$

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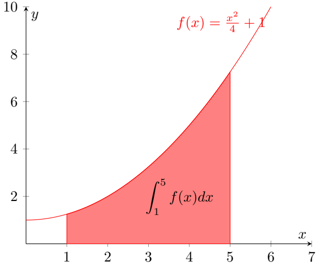


One way to approximate the area under a curve is by filling the region with rectangles and calculating the sum of the areas of the rectangles.

DEFINITION of left Sum

Given a function $f(x)$ and an interval $a \leq x \leq b$, $[a, b]$ and a positive integer n, define

Symbol: L_n

Spoken: the left sum with n rectangles.

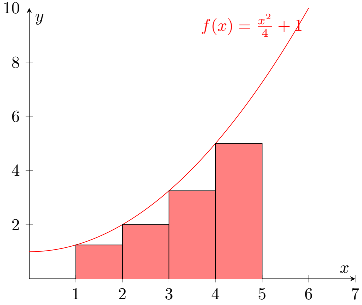
Meaning

- Subdivide the interval $[a, b]$ into n equal subintervals.
- On each subinterval, put a "left rectangle". That is, a rectangle that
  - Sits on the $x-$axis
  - Goes up or down until its left edge to touches the graph of f
- Add up the signed area of all the rectangles. The resulting sum is the value of L_n.

Analogous definition for the "Right sum", R_n, involving Right rectangles

Estimating Area

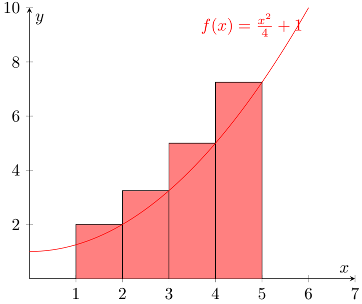
Take the width of each rectangle to be $\Delta x = 1$ ($\Delta x = \frac{b-a}{n} = \frac{5-1}{4} = 1$). Summing the areas of the left rectangles results in a left sum of four rectangles, denoted by L_4. If we use the left endpoints, the heights of the four rectangles are $f(1), f(2), f(3), f(4)$.



$$\begin{aligned} L_4 &= f(1)\Delta x + f(2)\Delta x + f(3)\Delta x + f(4)\Delta x \\ &= 1.25 + 2 + 3.25 + 5 \\ &= 11.5 \end{aligned}$$

Estimating Area

We can repeat this using the right side of each rectangle to determine the height. The heights of each of the four rectangles are now $f(2)$, $f(3)$, $f(4)$ and $f(5)$, respectively.



$$\begin{aligned} R_4 &= f(2)\Delta x + f(3)\Delta x + f(4)\Delta x + f(5)\Delta x \\ &= 2 + 3.25 + 5 + 7.25 \\ &= 17.5 \end{aligned}$$

Estimating Area

Observe: If $f(x)$ is increasing on the interval $[a,b]$, then the actual value of the (signed) Area will be the sandwiched in between L_n and R_n

$$L_n < \text{Area} = \int_a^b f(x)dx < R_n$$

Similarly, if $f(x)$ is decreasing on $[a,b]$, then

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$$R_n < \text{Area} = \int_a^b f(x)dx < L_n$$

In our example, we have

$$L_4 = 11.5 < \text{Area} = \int_1^5 f(x)dx < R_4 = 17.5$$

Note that the average of L_4 and R_4 would be an even better approximation: Area$\approx (11.5 + 17.5)/2 = 14.5$

Estimating Area

The previous average of 14.5 is very close to the actual area of $14.333....$

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The previous average of 14.5 is very close to the actual area of $14.333...$.
Our accuracy can be improved if we increase the number or rectangles, and let Δx get smaller.
The difference between the actual value of the definite integral and either the left or right Riemann sum is the error in that approximation.
But in both of the cases above, this error can be no larger than the (absolute value of the) difference between the left and right sums, since one is an underestimate and the other is an overestimate. Thus we have:

$$\mathrm{Error}_{\mathrm{L_n}} = |Area - L_n| \leq |R_n - L_n|$$
$$\mathrm{Error}_{\mathrm{R_n}} = |Area - R_n| \leq |R_n - L_n|$$

It is not hard to show that

$$|R_n - L_n| = |f(b) - f(a)|\Delta x$$

and that for n equal subintervals, $\Delta x = \frac{b-a}{n}$.

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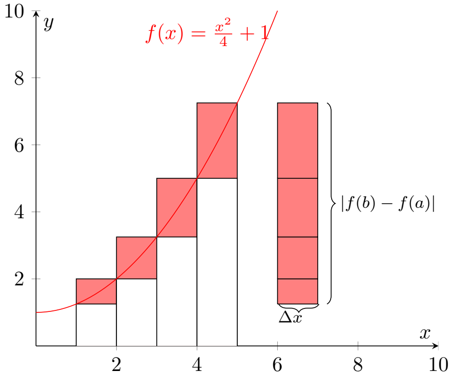
Difference between R_4 and L_4



THEOREM 1 Error Bounds for Approximations of Area by Left or Right Sums

If $f(x) > 0$ and is either increasing on $[a, b]$ or decreasing on $[a, b]$, then

$$|f(b) - f(a)|\frac{b-a}{n}$$

is an error bound for the approximation of the area, by L_n or R_n.

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For our previous example:

$$\text{Error} \leq |R_4 - L_4| = |f(5) - f(1)| \cdot 1 = 6$$

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THEOREM2 Limits of Left and Right Sums

If $f(x) > 0$ and is either increasing on $[a, b]$ or decreasing on $[a, b]$, then its left and right sums approach the same real number I as $n \to \infty$

This number I is the area between the graph of f and the x axis from $x = a$ to $x = b$.

Approximating Area

In our example, we had

$$\text{Error} \leq |7.25 - 1.25| \frac{5-1}{4} = 6$$

If we wanted a particular accuracy, say 0.05, we could use the error formula to calculate n, the number of rectangles needed:

$$|7.25 - 1.25| \frac{5-1}{n} = 0.05$$

Solving for n yields $n = 480$. We would need at least 480 rectangles to guarantee an accuracy of 0.05.

Definite Integral as Limit of Sums

We now come to a general definition of the definite integral.

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We now come to a general definition of the definite integral.
Let f be a function on interval $[a, b]$. Partition $[a, b]$ into n subintervals of equal length $\Delta x = \frac{b-a}{n}$ with endpoints

$$a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b$$

In each subinterval, choose an arbitrary point $x_{k-1} < c_k < x_k$
Then, using summation notation we have

$$L_n = f(x_0)\Delta x + f(x_1)\Delta x + \ldots + f(x_{n-1})\Delta x = \sum_{k=1}^{n} f(x_{k-1})\Delta x$$

$$R_n = f(x_1)\Delta x + f(x_2)\Delta x + \ldots + f(x_n)\Delta x = \sum_{k=1}^{n} f(x_k)\Delta x$$

$$S_n = f(c_1)\Delta x + f(c_2)\Delta x + \ldots + f(c_n)\Delta x = \sum_{k=1}^{n} f(c_k)\Delta x$$

S_n is called a Riemann sum. Notice that L_n and R_n are both special cases of a Riemann sum.

Area (Revisited)

Let's revisit our original problem and calculate the Riemann sum using the midpoints for c_k ($c_k = \frac{x_{k-1}+x_k}{2}$).

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The width of each rectangle is again $\Delta x = 1$. The heights are now $f(\frac{3}{2}), f(\frac{5}{2}), f(\frac{7}{2}), f(\frac{9}{2})$.
The sum of the rectangles is then

$$S_4 = 1.5625 + 2.5625 + 4.0625 + 6.0625 = 14.25$$

This is quite close to the actual area of $14.333...$

THEOREM 3 Limit of Riemann Sums

If f is a continuous function on $[a, b]$, then the Riemann sums for f on $[a, b]$ approach a real number limit I as $n \to \infty$.

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If f is a continuous function on $[a, b]$, then the Riemann sums for f on $[a, b]$ approach a real number limit I as $n \to \infty$.

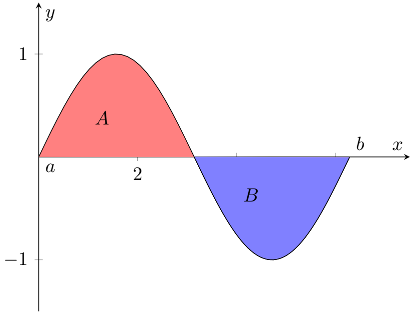
This limit I of the Riemann sums for f on $[a, b]$ is called the definite integral of f from a to b, denoted

$$\int_a^b f(x)dx$$

The integrand is $f(x)$, the lower limit of integration is a, and the upper limit of integration is b.

Negative Values

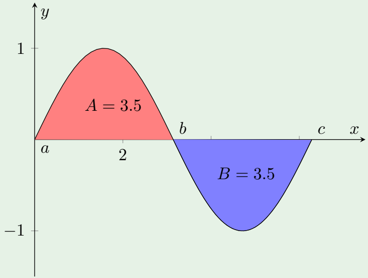
If $f(x)$ is positive for some values of x on $[a, b]$ and negative for others, then the definite integral symbol $\int_a^b f(x)dx$ represents the cumulative sum of the signed areas between the graph of $f(x)$ and the $x-$axis, where areas above are positive and areas below negative.



$$\int_a^b f(x)dx = A - B$$

Examples:

Calculate the definite integrals by referring to the figure with the indicated areas.



$$\int_a^a f(x)dx = 3.5$$
$$\int_b^c f(x)dx = -3.5$$
$$\int_a^c f(x)dx = 3.5 - 3.5 = 0$$

PROPERTIES **Properties of Definite Integrals**

1. $\int_a^a f(x)dx = 0$
2. $\int_a^b f(x)dx = -\int_b^a f(x)dx$
3. $\int_a^b kf(x)dx = k\int_a^b f(x)dx$
4. $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$
5. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

Exercises:

1. Calculate the indicated Riemann sum for the function $f(x) = x^2 + 5x$.
   - Right sum R_3 on $[0, 3]$.
   - Left sum L_3 on $[0, 3]$.
   - Partion $[0, 3]$ into 3 subintervals of equal length and let $c_k = \frac{x_{k-1}+x_k}{2}$
2. Compute the error bounds for R_3 and L_3 if $f(x) = x^2 + 5x$ on $[0, 3]$
3. Calculate the following definite integrals given that:

$$\int_0^6 x dx = 18, \quad \int_0^6 x^2 dx = 72 \quad \int_3^6 x^2 dx = 63.$$

$$a. \int_0^6 3x dx \quad b. \int_0^6 (x - x^2) dx \quad c. \int_0^3 x^2 dx \quad \int_6^3 x^2 dx$$

4. How large must n be chosen so that $\int_0^3 (x^2 + 5x) dx = R_n \pm 0.1$?

6. Integration

Learning Objectives

- Use the Fundamental Theorem of Calculus to evaluate definite integrals.
- Find the average value of a function.

THEOREM 1 Fundamental Theorem of Calculus

If f is a continuous function on the closed interval $[a, b]$, and F is any antiderivative of f, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

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Evaluating Definite Integrals

By the fundamental theorem we can evaluate

$$\int_a^b f(x)dx$$

easily and exactly. We simply calculate

$$F(b) - F(a)$$

PROPERTIES **Properties of Definite Integrals**

1. $\int_a^a f(x)dx = 0$
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3. $\int_a^b kf(x)dx = k\int_a^b f(x)dx$
4. $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$
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Example 1:

$$\int_1^3 5dx = 5\int_1^3 dx = 5x\Big|_1^3 = 5[3-1] = 5\cdot 2 = 10$$

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Example 3:

$$\int_0^3 x^2 dx = \frac{x^3}{3}\Big|_0^3 = \frac{3^3}{3} - \frac{0^3}{3} = 9 - 0 = 9$$

Example 4: (Definite Integrals and Substitution)

$$\int_{-1}^{1} e^{2x}dx = \frac{1}{2}\int_{-2}^{2} e^{u}du = \frac{e^u}{2}\Big|_{-2}^{2} = \frac{e^2}{2} - \frac{e^{-2}}{2}$$

If $u = 2x,$ then $du = 2dx,$ and

$x = -1$ implies $u = -2$

$x = 1$ implies $u = 2$

Example 5:

$$\int_{1}^{2} \frac{1}{x}dx = \ln x\Big|_{1}^{2} = \ln 2 - \ln 1 = \ln 2$$

Example 6: (Definite Integrals and Substitution)

$$\int_0^5 \frac{x^2}{x^3+4}dx = \frac{1}{3}\int_4^{129} \frac{1}{u}du = \frac{\ln u}{3}\bigg|_4^{129} = \frac{\ln 129}{3} - \frac{\ln 4}{3}$$

Let $u = x^3 + 4,$ then $du = 3x^2 dx,$ and

$x = 0$ implies $u = 4$

$x = 5$ implies $u = 129$

Example 7:

From past records a management service determined that the rate of increase in maintenance cost for an apartment building (in dollars per year) is given by

$$M'(x) = 90x^2 + 5,000,$$

where $M(x)$ is the total accumulated cost of maintenance for x years.
Write a definite integral that will give the total maintenance cost from the end of the second year to the end of the seventh year. Evaluate the integral.

Example 7:

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Write a definite integral that will give the total maintenance cost from the end of the second year to the end of the seventh year. Evaluate the integral.

$$\int_3^7 (90x^2 + 5,000)dx = 30x^3 + 5000x\Big|_3^7 = 12,290 + 35000 - 810 - 15,000 = \$29,480$$

Using Definite Integrals for Average Values

The average value of a continuous function f over $[a, b]$ is

$$\frac{1}{b-a}\int_a^b f(x)dx$$

Note this is the area under the curve divided by the width.
Hence, the result is the average height or average value.

Example:

The total cost (in dollars) of printing x dictionaries is

$$C(x) = 20,000 + 10x$$

a. Find the average cost per unit if 1000 dictionaries are produced.
b. Find the average value of the cost function over the interval $[0, 1000]$.
c. Write a description of the difference between part a. and part b.

Example:

a. The average cost is

$$\bar{C}(x) = \frac{C(x)}{x} = \frac{20000}{x} + 10$$

$$\bar{C}(1000) = \frac{20000}{1000} + 10 = 30$$

b. Find the average value of the cost function over the interval $[0, 1000]$.

$$\begin{aligned}\frac{1}{b-a}\int_a^b C(x)dx &= \frac{1}{1000}\int_0^{1000}(20,000 + 10x)dx \\ &= \frac{1}{1000}(20,000x + 5x^2)\Big|_0^{1000} \\ &= \frac{1}{1000}(20,000 \cdot 1000 + 5(1000)^2) \\ &= 20,000 + 5,000 = 25,000\end{aligned}$$

c. Write a description of the difference between part a. and part b.

Example:

c. If you just do the set-up for printing, it costs $\$20,000$. This is the cost for printing 0 dictionaries.

   If you print $1,000$ dictionaries, it costs $\$30,000$. That is $\$30$ per dictionary (part a).

   If you print some random number of dictionaries (between 0 and 1000), on average it costs $\$25,000$ (part b).
   Those two numbers really have not much to do with one another.

Exercises:

1. Evaluate the following definite integrals.

$$a.\ \int_0^4 x dx, \quad b.\ \int_{-1}^1 (1-x^3)dx, \quad c.\ \int_0^1 e^{7x}dx, \quad d.\ \int_1^5 \frac{x+2}{x^2+4x}dx$$

2. Find the average value of :

$$a.\ g(x) = x^2 + 4 \text{ on } [-2,2] \qquad b.\ h(x) = \sqrt{x} \text{ on } [0,9]$$

3. The total cost (in dollars) of making x music boxes is given by $C(x) = 12,000 + 40x$
   a. Find the average cost per unit if 200 music boxes are produced.
   b. Find the average value of the cost function on the interval $[0, 200]$.
   c. Explain the difference in the meaning of the values found in parts a and b.

4. A company produces a printer that also scans documents. The research department produced the marginal cost function $C'(x) = 200 - \frac{x}{5}$ where $C(x)$ is the total cost (in dollars) and x is the number of printers produced in a month. Compute the increase in cost going from a production level of 100 printers per month to 500 printers per month. Set up a definite integral and evaluate.

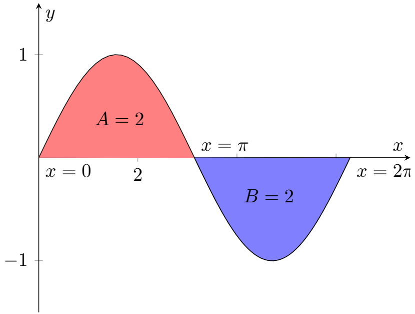
6. Integration

Learning Objectives

- Determine the area of a region bound by two or more curves.
- Solve applications pertaining to income distribution.

Negative Values for the Definite Integral

Remember the idea of unsigned area and signed area.



- Usigned Area between graph of $f(x)$ and the $x-$axis between $x = 0$ and $x = 2\pi$ is $\mathrm{USA} = A + B = 2 + 2 = 4$.
- Signed Area is $\mathrm{SA} = A - B = 2 - 2 = 0 = \int_{x=0}^{x=2\pi} f(x)dx = \text{the definite integral.}$

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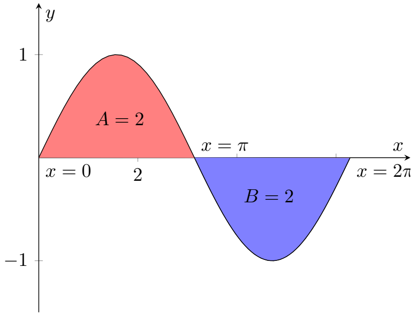

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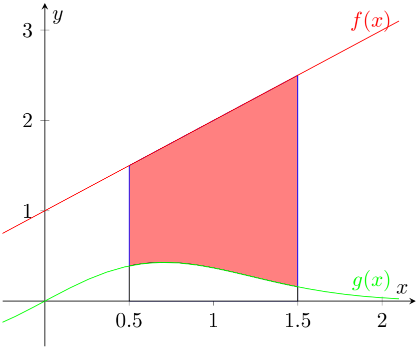
In this sec we'll study the "area betwen curves" these words mean the unsigned area.

THEOREM 1 Area Between Two Curves

If $f(x) \geq g(x)$ on some interval $[a, b]$, then (unsigned) area bounded by $y = f(x)$ and $y = g(x)$ for $a \leq x \leq b$ (in other words the area between the graph of f and g from $x = a$ to $x = b$) is given by

$$\int_{x=a}^{x=b} [f(x) - g(x)]dx$$



Example:

Find the area bounded by

$$y = x^2 - 1 \text{ and}$$
$$y = 3.$$

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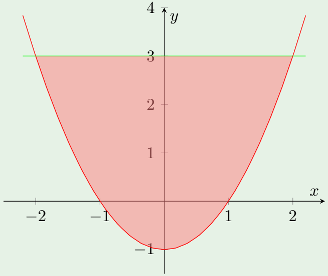
Note the two curves intersect at $x = \pm 2$, and $y = 3$ is the larger function on $[-2, 2]$.



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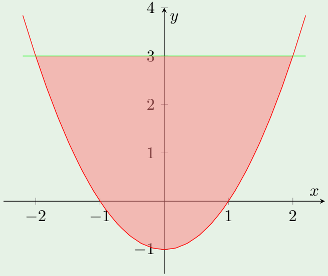

$$\int_{-2}^{2} [3 - (x^2 - 1)] dx = \int_{-2}^{2} [4 - x^2] dx = 4x - \frac{x^3}{3} \Big|_{-2}^{2} = 4(2) - \frac{2^3}{3} - 4(-2) + \frac{-2^3}{3} = \frac{32}{3}$$

Example:

Find the area between the curves

$$y = x^2 + 1 \text{ and}$$
$$y = 2x - 2.$$

from $x = -3$ to $x = 2$.

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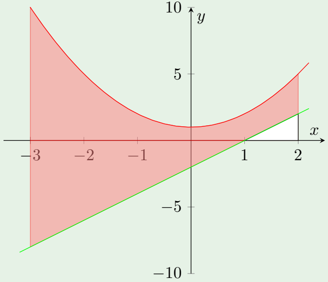

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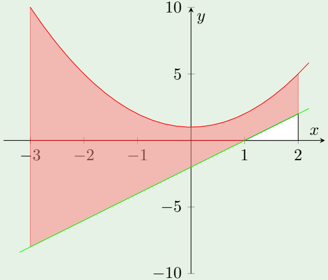
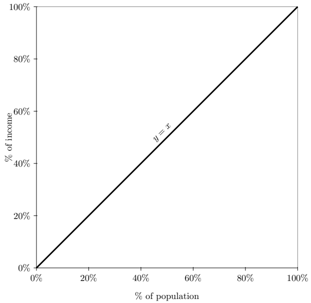


$$\int_{-3}^{2} [(x^2+1) - (2x-2)]dx = \int_{-3}^{2} (x^2 - 2x + 3)dx = \left(\frac{x^3}{3} - x + 3x\right)\Big|_{-3}^{2} = \frac{95}{3}$$

Application: Income Distribution

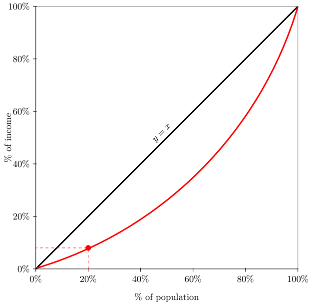
The quantity of interest is the yearly income per family.
One hypothetical extreme: every family has the same yearly income. "absolute equality".
Consider making a graph
$x-$axis: the percentage of the families in the country that are included.
$y-$axis: the percentage of the country's total yearly income that those families earn.



Application: Income Distribution

Now suppose a different scenario: some families make more money per year than others.

$x-$axis: the $\%$ of the families in the country that are included, with the lowest income family included first.

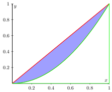


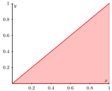
Data point $(20, 8)$ indicates that the bottom 20% of families receive only 8% of the total income

Gini Index

A measure of the inequality of income distribution is

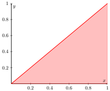
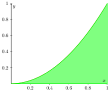
$$\frac{\text{area between the curves}}{\text{the area the red curve}} =$$





$$= \text{The Gini Index}$$

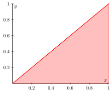
$$\text{area between the curves} = \int_{x=0}^{x=1} (x - f(x))dx = \int_{x=0}^{x=1} x dx - \int_{x=0}^{x=1} f(x)dx$$

$$=$$  $$-$$ 

A measure of 0 indicates absolute equality. A measure of 1 indicates absolute inequality (one family has all the income, the rest have none).

Gini Index

But the area of the red triangle is

$$A = \frac{1}{2} b \cdot h = \frac{1}{2}(1)(1) = \frac{1}{2} =$$ 

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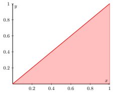
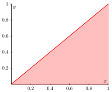


So the area between the curves is

$$\frac{1}{2} - \int_{x=0}^{x=1} f(x)dx =$$



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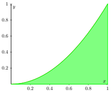

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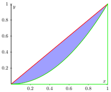

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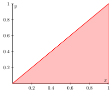

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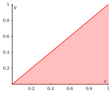

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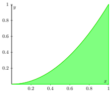

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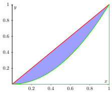

So the Gini Index will be:

$$G = \frac{\text{area between the curves}}{\text{the area the red curve}} = \frac{\frac{1}{2} - \int_{x=0}^{x=1} f(x)dx}{\frac{1}{2}} = 2\left(\frac{1}{2} - \int_{x=0}^{x=1} f(x)dx\right)$$

$$= 1 - 2\int_{x=0}^{x=1} f(x)dx$$

Example 1:

Suppose the curve of income distribution is given by $f(x) = x^8$. Find the Gini Index and draw a picture.

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$$G = 1 - 2\int_{x=0}^{x=1} f(x)dx = 1 - 2\int_{x=0}^{x=1} x^8 dx = 1 - 2(\frac{x^9}{9}\Big|_{x=0}^{x=1})$$
$$= 1 - 2\left(\frac{1^9}{9} - \frac{0^9}{9}\right) = 1 - \frac{2}{9} = \frac{7}{9}$$

Example 2:

A country is planning changes in tax structure in order to provide a more equitable distribution of income. The two Lorenz curves are:

$$f(x) = x^{2.3} \text{ currently, and}$$
$$g(x) = 0.4x + 0.6x^2 \text{ proposed.}$$

Will the proposed changes work?

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Will the proposed changes work?
Currently: Gini Index of income concentration=

$$1 - 2\int_{x=0}^{x=1} x^{2.3} dx = 1 - 2(\frac{x^{3.3}}{3.3}\Big|_{x=0}^{x=1}) = 1 - 2\left(\frac{1^{3.3}}{3.3} - \frac{0^{3.3}}{3.3}\right) = 1 - \frac{2}{3.3} \approx 0.3939$$

Future: Gini Index of income concentration=

$$1 - 2\int_0^1 0.4x + 0.6x^2 dx = 1 - 2(0.4\frac{x^2}{2} + 0.6\frac{x^3}{3}\Big|_0^1) = 1 - 2(0.2x^2 + 0.2x^3\Big|_0^1) = 0.20$$

The Gini index is decreasing, so the future distribution will be more equitable.

Exercises:

1. Find the area bounded by the graphs of the indicated equations over the given interval. Compute answers to three decimal places when necessary.

a. $y = 2x - 1, y = 5, -2 \leq x \leq 2$

b. $y = 3 - x^2, y = x - 2, -1 \leq x \leq 1$

c. $y = e^{2x}, y = 2 - x, 1 \leq x \leq 3$

2. Find the area bounded by the graphs of the indicated equations over the given intervals (when stated). Compute answers to three decimal places when necessary.

a. $y = -x^2 + 6, y = x$

b. $y = x^3 - 3x^2, y = x^2$

c. $y = x^4 - 3x^2, y = 7x^2 - 9$

3. Suppose that the following Lorenz curves represent the distribution of income for a small country in 2005 and 2008, respectively: $f(x) = x^2 + 0.4x - 0.5$, $g(x) = 0.25x^2 + 0.6x + 0.01$. Find the Gini index of the income concentration for both Lorenz curves and interpret the results.