Supplementary material for the paper: Root−n consistent estimation of the marginal density in semiparametric autoregressive time series models

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6 Proof of assertions (13) and (14) in the proof of Theorem 1

6.1 Proof of assertion (13)

We set for $j \in \mathbb{N}$,

$$\mathcal{H}_{nj} = \sigma \left( \varepsilon_s, \varepsilon^{(s)} : s \geq n' - j \right).$$

For this part we set

$$Z_i(s, v) = \frac{1}{\sigma_i} K_b \left[ \frac{v - m_i}{\sigma_i} - s \right],$$

and

$$Z_i(s, v) = Z_i(s, v) - \mathbb{E} \left[ Z_i(s, v) \right]$$

$$S_{v,ij} = A_{v,i(n'-j)} - \mathbb{E} \left( A_{v,i(n'-j)} \right) + \mathbb{E} \left[ A_{v,i(n'-j)} \right] \mathcal{H}_{n-1}.$$

Using the independence properties, observe that

$$S_{v,ij} = A_{v,i(n'-j)} - \frac{1}{\sigma_i} \int K(w) f_{\varepsilon} \left[ \frac{v - m_i}{\sigma_i} - bw \right] dw$$

$$- \int K(h) g_v (\varepsilon_{n'-j} + bh) \, dh + \mathbb{E} \left[ A_{v,i(n'-j)} \right].$$

From the first writing of $S_{v,ij}$, we have for $v, \bar{v} \in I$,

$$\mathbb{E} \left| \sum_{j=\ell}^{n'-1} \sum_{i=n'-j-\ell}^{n'} \left[ S_{v,ij} - S_{v,ij} \right] \right|^2 \leq 2 \sum_{j=\ell}^{n'-1} \mathbb{E} \left| \sum_{i=n'-j-\ell}^{n'} \left[ Z_i(s, v) - Z_i(s, \bar{v}) \right] \right|^2 f_{\varepsilon}(s) ds.$$ 

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Moreover, for $s \in \mathbb{R}$, we have, using $\ell$–dependence,

$$
E \left| \sum_{i=n' - j - \ell}^{n'} [Z_i(s, v) - Z_i(s, \bar{v})] \right|^2 \leq \ell \sum_{h=1}^{k-1} \sum_{g=0}^{\ell} E |Z_{h+g\ell}(s, v) - Z_{h+g\ell}(s, \bar{v})|^2.
$$

Moreover, we have

$$
E \int |Z_i(s, v) - Z_i(s, \bar{v})|^2 f_\varepsilon(s) ds \leq 2E \int \left| \frac{1}{\sigma_i} K_b \left[ \frac{v - m_i}{\sigma_i} - s \right] - \frac{1}{\sigma_i} K_b \left[ \frac{\bar{v} - m_i}{\sigma_i} - s \right] \right|^2 f_\varepsilon(s) ds
$$

$$
\leq \frac{C |v - \bar{v}|^{\delta_1}}{b^{1+\delta_1}} \int \left| K(z) - K \left( \frac{\bar{v} - v}{b\sigma_i} + z \right) \right|^{2-\delta_1} f_\varepsilon \left( \frac{v - m_i}{\sigma_i} - bz \right) dz
$$

$$
\leq \frac{C |v - \bar{v}|^{\delta_1}}{b^{1+\delta_1}} \int \int 1_{w \in [z, z + \frac{\bar{v} - v}{b\sigma_i}]} |K'(w)|^{2-\delta_1} dw dz
$$

$$
\leq \frac{C}{b^{2+\delta_1}} |v - \bar{v}|^{1+\delta_1}.
$$

Then we get,

$$
\frac{1}{n^3} \left| \sum_{j=\ell}^{n-1} \sum_{i=n' - j - \ell}^{n'} [S_{v,ij} - S_{\bar{v},ij}] \right|^2 \leq \frac{C\ell}{nb^{2+\delta_1}} |v - \bar{v}|^{1+\delta_1}.
$$

Note that $\frac{\ell}{nb^{2+\delta_1}} \rightarrow 0$. Moreover, it is easily seen that

$$
\frac{1}{n^3} \left| \sum_{j=\ell}^{n-1} \sum_{i=n - j - \ell}^{n} S_{v,ij} \right|^2 \leq \frac{C\ell}{nb^2} = o(1).
$$

This means that $G_n : v \mapsto \frac{1}{n^{3/2}} \sum_{j=\ell}^{n-1} \sum_{i=n - j - \ell}^{n} S_{v,ij}$ converges pointwise to 0 in probability. Note that $G_n$ is a random function taking values in $C(I)$, the space of real-valued and continuous functions defined on the compact interval $I$. From (20) and the Kolmogorov-Chentsov tightness criterion in $C(I)$ (see for instance Kallenberg (1997), Corollary 14.9), we (13).

### 6.2 Proof of assertion (14)

We have to prove the following convergence in probability:

$$
\frac{1}{n^{3/2}} \sup_{v \in I} \left| \sum_{(i,j) \in \mathcal{I}_n} \int K(h) [g_v(\varepsilon_j + bh) - g_v(\varepsilon_j)] dh \right| = o_P(1).
$$

The proof is divided into two parts. In the first part, we show that

$$
\frac{1}{n^{3/2}} \sup_{v \in I} \left| \sum_{(i,j) \in \mathcal{I}_n} \int K(h)E [g_v(\varepsilon_j + bh) - g_v(\varepsilon_j)] dh \right| = o_P(1).
$$
We have
\[
\int K(h)E [g_v(\varepsilon_j + bh) - g_v(\varepsilon_j)] dh = \int \int K(h) [g_v(u + bh) - g_v(u)] f_\varepsilon(u) du dh \\
= \int \int K(h)g_v(z) [f_\varepsilon(z - bh) - f_\varepsilon(z)] dh dz.
\]

Using assertion (11) given in the paper, the condition \(\sqrt{n}b^2 \to 0\) and the fact that \(\sup_{v \in I} \int g_v(z) dz \leq C\), we get (22). To show (21), it remains to show that
\[
\frac{1}{n^{3/2}} \sup_{v \in I} \left| \sum_{(i,j) \in \mathcal{I}_n} \int K(h) \left[ g_v(\varepsilon_j + bh) - g_v(\varepsilon_j) \right] dh \right| \\
= \frac{1}{\sqrt{n}} \sup_{v \in I} \sum_{j=1}^{n'} c_{n,j} \int K(h) \left[ g_v(\varepsilon_j + bh) - g_v(\varepsilon_j) \right] dh \\
= o_p(1),
\]
where \(c_{n,j} = \frac{1}{n} \sum_{i=1}^{n'} 1_{|i-j| \geq \ell}\).

1. To show (23) when \(I\) is the singleton \(\{v\}\), we use Jensen inequality and the fact that the translations are continuous in \(L^2(f_\varepsilon)\) for any function in \(L^2(\mu)\). More precisely, we have
\[
n^{-3/2} \left| \sum_{(i,j) \in \mathcal{I}_n} \int K(h) \left[ g_v(\varepsilon_j + bh) - g_v(\varepsilon_j) \right] dh \right|^2 \\
\leq n^{-1} \sum_{j=1}^{n'} c_{n,j}^2 \int \left| \int K(h) [g_v(u + bh) - g_v(u)] dh \right|^2 f_\varepsilon(u) du \\
\leq n'/n \int \int K(h) [g_v(u + bh) - g_v(u)]^2 f_\varepsilon(u) du dh \\
\leq \sup_{|z| \leq b} \int [g_v(u + z) - g_v(u)]^2 f_\varepsilon(u) du \\
= o_p(1).
\]

2. To show (23) when \(I\) is a compact interval not reduced to one point, we proceed in two parts.
   (a) We first show that
\[
\frac{1}{\sqrt{n}} \sup_{v \in I} \left| \sum_{j=1}^{n'} c_{n,j} \int K(h) \left[ g_v(\varepsilon_j + bh) - g_v(\varepsilon_j) \right] dh \right| = o_p(1). \tag{24}
\]
To this end, we will use Lemma 19.34 in van der Vaart (1998). We set
\[
g_{n,v}(x) = \int K(h) [g_v(x + bh) - g_v(x)] dh.
\]
For the family \( \mathcal{G}_{n,I} = \{ g_{n,v} : v \in I \} \) of functions, we consider the envelope function \( G_n \) defined by
\[
G_n(x) = \int K(h) [G(x + bh) + G(x)] \, dh,
\]
where \( G \) is defined in assumption A7. For bounding the bracketing numbers of this family, we first observe that if \([f_1, f_2] \) is an \( \epsilon \)-bracketing in \( \mathcal{G}_I \), then \( f_1 \leq g_\epsilon \leq f_2 \) (i.e \( \int (f_2 - f_1)^2 \, d\mu < \epsilon^2 \) entails that
\[
f_{n,1}(x) = \int K(h) [f_1(x + bh) - f_2(x)] \, dh
\]
\[
\leq g_{n,v}(x) \leq \int K(h) [f_2(x + bh) - f_1(x)] \, dh = f_{n,2}(x).
\]
Moreover, one can show that
\[
\int \left| f_{n,2}(x) - f_{n,1}(x) \right|^2 f_\epsilon(x) \, dx \leq 4\epsilon^2.
\]
Then, we have
\[
N_{\|} \left( \sqrt{2}\epsilon, \mathcal{G}_{n,I}, \mathbb{L}^2(\epsilon_0) \right) \leq N_{\|} (\epsilon, \mathcal{G}_I, \mathbb{L}^2(\mu))
\]
and the bracketing numbers of the family \( \mathcal{G}_{n,I} \) are of polynomial decay. Next we show that
\[
\sup_{v \in I} \left\| \int g_{n,v}(x)^2 f_\epsilon(x) \, dx \right\| = o_\sharp(1). \tag{25}
\]
To show (25), we consider for a given \( \epsilon > 0 \), some brackets \( I_1, I_2, \ldots, I_T \) that cover \( \mathcal{G}_I \). For each integer \( 1 \leq p \leq T \), we consider an element \( g_{v_p} \in I_p \). Then, for \( 1 \leq p \leq T \), if \( I_p = [f^{(p)}_1, f^{(p)}_2] \), we set \( I_{n,p} = [f^{(p)}_{n,1}, f^{(p)}_{n,2}] \), with
\[
f^{(p)}_{n,1}(x) = \int K(h) \left[ f^{(p)}_1(x + bh) - f^{(p)}_2(x) \right] \, dh,
\]
\[
f^{(p)}_{n,2}(x) = \int K(h) \left[ f^{(p)}_2(x + bh) - f^{(p)}_1(x) \right] \, dh.
\]
Then, if \( g_v \in I_p \), we have \( g_{n,v} \in I_{n,p} \) and
\[
\int g_{n,v}^2(x) f_\epsilon(x) \, dx \leq 2 \int \left[ g_{n,v}(x) - g_{n,v_p}(x) \right]^2 f_\epsilon(x) \, dx + 2 \int g_{n,v_p}(x) f_\epsilon(x) \, dx
\]
\[
\leq 4\epsilon^2 + 2 \max_{1 \leq p \leq T} \int g_{n,v_p}^2(x) f_\epsilon(x) \, dx.
\]
Since for each \( v \in I \), we have \( \int g_{n,v}(x)^2 f_\epsilon(x) \, dx = o(1) \), we conclude that
\[
\limsup_{n \to \infty} \int g_{n,v}^2(x) f_\epsilon(x) \, dx \leq 4\epsilon^2.
\]
Since \( \epsilon \) is arbitrary, we conclude (25).
Moreover, setting, for $\kappa > 0$, $a_n(\kappa) = \kappa/\text{Log} \left[ N_\parallel (\kappa, G_{n,1}, \mathbb{L}^2(\varepsilon_0)) \right]$, we have

$$\sqrt{n} \mathbb{E} G_n(\varepsilon_0) \mathbb{I}_{G_n(\varepsilon_0) > \sqrt{n} a_n(\kappa)} \leq \frac{\sqrt{n}}{(\sqrt{n} a_n(\kappa))^{1+\delta}} \mathbb{E} G_n(\varepsilon_0)^{2+\delta}$$

$$\leq \frac{2}{n^{\delta/2} a_n(\kappa)^{1+\delta}} \int G(x)^{2+\delta} d\mu(x).$$

Here $\text{Log}(x) = \log(x) \wedge 1$. Then, from (25), one can choose $\kappa^2 = \kappa_0^2 \geq \int g_{n,v}(x)^2 f_\varepsilon(x) dx$ such that $\kappa \to 0$ and $n^{\delta/2} a_n(\kappa)^{1+\delta} \to \infty$. Hence, from Lemma 19.34 in van der Vaart (1998), we deduce that

$$\frac{1}{\sqrt{n}} \mathbb{E} \sup_{v \in I} \left| \sum_{j=1}^{n'} g_{n,v}(\varepsilon_j) \right| = o_p(1)$$

and hence (24).

(b) Finally, we have

$$\sup_{v \in I} \left| n^{-3/2} \sum_{(i,j) \in I_n} g_{n,v}(\varepsilon_j) - \frac{1}{\sqrt{n}} \sum_{j=1}^{n'} g_{n,v}(\varepsilon_j) \right|$$

$$\leq \frac{1}{\sqrt{n}} \sum_{j=1}^{n'} |c_{n,j} - 1| \int K(h) |G(\varepsilon_j + bh) + G(\varepsilon_j)| dh$$

$$\leq \frac{1}{n} \sum_{j=1}^{n'} \int K(h) |G(\varepsilon_j + bh) + G(\varepsilon_j)| dh \cdot \frac{n - n' + 2\ell - 1}{\sqrt{n}}.$$

Using the fact that $\ell \sim n^t$ with $t < 1/2$, we deduce from assumption A7 that

$$\sup_{v \in I} \left| n^{-3/2} \sum_{(i,j) \in I_n} g_{n,v}(\varepsilon_j) - \frac{1}{\sqrt{n}} \sum_{j=1}^{n'} g_{n,v}(\varepsilon_j) \right| = o_P(1).$$

From the last convergence and from (24), we deduce (23).

7 Proof of Corollary 1

From Theorem 1, Theorem 2 and assumption A8, we have

$$\sqrt{n} \left[ f_X(v) - f_X(v) \right] = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} M_{i,v} + o_p(1).$$

The first part of the corollary concerns the convergence of the finite dimensional distributions. Note also that the previous convergence is uniform if assumption A7 holds true. Convergence of finite dimensional distributions is straightforward using a central limit theorem for weakly dependent time series. For instance, the central limit theorem given in Zhao (2010), Theorem 3, applies in our case. For the uniform convergence, it remains to show the tightness of the empirical process.
\( G_n : v \mapsto \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \overline{M}_{i,v} \). Since we have assumed \( A7 \), it is only necessary to study the tightness in \( C(I) \), the space of real-valued and continuous function defined on \( I \), of
\[
\tilde{G}_n : v \mapsto \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left[ \frac{1}{\sigma_i} \hat{f}_{\varepsilon} \left( \frac{v - m_i}{\sigma_i} \right) - \mathbb{E} \frac{1}{\sigma_i} \hat{f}_{\varepsilon} \left( \frac{v - m_i}{\sigma_i} \right) \right].
\]

To this end, we use the Kolmogorov-Chentsov criterion (see for instance Kallenberg (1997), Corollary 14.9). If \( v, \bar{v} \in I \) satisfy \( v \leq \bar{v} \), we have from Jensen inequality,
\[
\mathbb{E} \left| \tilde{G}_n(v) - \tilde{G}_n(\bar{v}) \right|^2 \leq |v - \bar{v}| \int \mathbb{E} \left| \tilde{G}_n'(u) \right|^2 du \\
\leq |v - \bar{v}|^2 \sup_{u \in I} \mathbb{E} \left| \tilde{G}_n'(u) \right|^2.
\]

Then the tightness will follow if we show that
\[
\sup_{u \in I} \mathbb{E} \left| \tilde{G}_n'(u) \right|^2 = O(1). \tag{26}
\]

Note that
\[
\tilde{G}_n'(u) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left[ \frac{1}{\sigma_i^2} \hat{f}_{\varepsilon} \left( \frac{u - m_i}{\sigma_i} \right) - \mathbb{E} \frac{1}{\sigma_i^2} \hat{f}_{\varepsilon} \left( \frac{u - m_i}{\sigma_i} \right) \right].
\]

Moreover, for \( i \leq j, v \in I \) and \( \ell = j - i \), we have
\[
\text{Cov} \left[ \frac{1}{\sigma_i^2} \hat{f}_{\varepsilon} \left( \frac{v - m_i}{\sigma_i} \right), \frac{1}{\sigma_j^2} \hat{f}_{\varepsilon} \left( \frac{v - m_j}{\sigma_j} \right) \right] \\
\leq \frac{\| \hat{f}_{\varepsilon} \|_\infty}{\gamma^2} \mathbb{E} \left| \frac{1}{\sigma_j^2} \hat{f}_{\varepsilon} \left( \frac{v - m_j}{\sigma_j} \right) - \frac{1}{\sigma_{j\ell}^2} \hat{f}_{\varepsilon} \left( \frac{v - m_{j\ell}}{\sigma_{j\ell}} \right) \right|.
\]

Then using assumption \( A2 \) and assumption \( A5 \), we deduce that there exist \( C > 0 \) such that for all \( i \leq j \),
\[
\text{Cov} \left[ \frac{1}{\sigma_i^2} \hat{f}_{\varepsilon} \left( \frac{v - m_i}{\sigma_i} \right), \frac{1}{\sigma_j^2} \hat{f}_{\varepsilon} \left( \frac{v - m_j}{\sigma_j} \right) \right] \leq Ca^{\frac{i-j}{2}},
\]
where \( a \in (0,1) \) is defined in assumption \( A2 \). The proof of the last inequality uses the same arguments than the proof of Lemma 6. This control of covariances immediately implies (26). The tightness criterion of Kolmogorov-Chentsov applies. The proof of Corollary 1 is now complete. □

8 Proofs of the results of Subsection 5.6

In the subsequent proofs, \( C > 0 \) will denote a generic constant that can change from line to line. Moreover, if \( X \) is a random variable, we set \( X = X - \mathbb{E}(X) \).

We first observe that if the kernel \( K \) is Lipschitz continuous and bounded, we have for any \( q \in (0,1) \),
\[
|K(x) - K(y)| \leq (2\|K\|_\infty)^{1-q} \text{Lip}(K)^q |x - y|^q.
\]
In particular, \( K \) is Hölder continuous with exponent \( q \). This fact will be used in the proof of the lemmas 1 and 2.
8.1 Proof of Lemma 1

Since \( \sqrt{n} \left( \hat{\theta} - \theta_0 \right) = O_p(1) \), it is enough to prove that

\[
\frac{1}{n^2} \sum_{i,j=1}^{n} \sup_{v \in I, \theta \in \Theta_{0,s}} \left| \frac{1}{\sigma_i(\theta)} K_b [L_{v,ij}(\theta)] - \frac{1}{\sigma_{i\ell}(\theta)} K_b [\overline{L}_{v,ij}(\theta)] \right| = o_p \left( n^{-1/2} \right),
\]

where \( \epsilon > 0 \) is defined in assumption A2. Let \( q \) be a positive real number such that \( 2q \leq \min(\delta, s) \).

Using assumption A2, we have

\[
\max_{\theta \in \Theta} \left| \frac{1}{\sigma_i(\theta)} - \frac{1}{\sigma^2_i(\theta)} \right| \leq C \max_{\theta \in \Theta} |\sigma_i^2(\theta) - \sigma^2_i(\theta)|^q.
\]

One can choose for instance \( C = 2^{-q} \gamma^{-1-2q} \). Moreover, setting \( q = s/2 \), we have

\[
\left| K_b [L_{v,ij}(\theta)] - K_b [\overline{L}_{v,ij}(\theta)] \right| \\
\leq C b^{-1-q} |L_{v,ij}(\theta) - \overline{L}_{v,ij}(\theta)|^q \\
\leq C b^{-1-q} \left( |m_i(\theta) - \overline{m}_i(\theta)|^q + |m_i(\theta)|^q \cdot |\sigma_i^2(\theta) - \sigma_i^2(\theta)|^q \right) \\
+ C b^{-1-q} \left( |m_j(\theta) - \overline{m}_j(\theta)|^q + |\sigma_j^2(\theta) - \sigma_j^2(\theta)|^q \cdot |X_j - m_j(\theta)|^q \right).
\]

Using assumption A2 and the Cauchy-Schwarz inequality, we deduce that

\[
\sup_{v \in I} \left\| \hat{f}(v) - \overline{f}(v) \right\| = O_p \left( \frac{1}{nb^{1+q}} \sum_{i=1}^{n} a_{v,i}^q \right) = O_p \left( \frac{1}{nb^{1+q}} \right). \]

8.2 Proof of Lemma 2

Since \( \sqrt{n} \left( \hat{\theta} - \theta_0 \right) = O_p(1) \), it is enough to prove that

\[
\frac{1}{n^2} \sum_{i,j=1}^{n} \sup_{v \in I, \theta \in \Theta_{0,s}} \left| \frac{1}{\sigma_i(\theta)} K_b [L_{v,ij}(\theta)] - \frac{1}{\sigma_{i\ell}(\theta)} K_b [L_{v,ij\ell}(\theta)] \right| = o_p \left( n^{-1/2} \right),
\]

where \( \epsilon > 0 \) is defined in assumption A2.

- As in the proof of Lemma 1, we use assumption A2 to get
  \[
  \left| \frac{1}{\sigma_i(\theta)} - \frac{1}{\sigma_{i\ell}(\theta)} \right| \leq C \max_{\theta \in \Theta} |\sigma_i^2(\theta) - \sigma_{i\ell}^2(\theta)|^q.
  \]

- Using the previous point, we have also for \( q = s/2 \),

\[
\sup_{v \in I} |K_b [L_{v,ij}(\theta)] - K_b [L_{v,ij\ell}(\theta)]| \\
\leq C b^{-1-q} \sup_{v \in I} |L_{v,ij}(\theta) - L_{v,ij\ell}(\theta)|^q \\
\leq C b^{-1-q} \left( |m_i(\theta) - m_{i\ell}(\theta)|^q + |m_i(\theta)|^q \cdot |\sigma_i^2(\theta) - \sigma_{i\ell}^2(\theta)|^q \right) \\
+ C b^{-1-q} \left( |m_j(\theta) - m_{j\ell}(\theta)|^q + |X_j - X_{j\ell}|^q + |\sigma_j^2(\theta) - \sigma_{j\ell}^2(\theta)|^q \cdot |X_j - m_j(\theta)|^q \right).
\]
Using assumption A2, we get
\[ \mathbb{E} \sup_{\theta \in \Theta_0, v \in \ell} |K_b[\mathcal{L}_{v,ij}(\theta)] - K_b[\mathcal{L}_{v,ij\ell}(\theta)]| \leq C_{b^{1+q}}. \]

Using the two previous points and assumption A2, we get
\[ \frac{1}{n^2} \sum_{i,j=1}^{n} \sup_{v \in \ell, \theta \in \Theta_0, \epsilon} \left\| \frac{1}{\sigma_i(\theta)} K_b[\mathcal{L}_{v,ij}(\theta)] - \frac{1}{\sigma_{ij}(\theta)} K_b[\mathcal{L}_{v,ij\ell}(\theta)] \right\| = O_{\mathbb{P}} \left( \frac{a_{b^{1+q}}}{b^{1+q}} \right). \]

Then the result follows from the conditions \( \ell \sim n^t, 0 < a < 1 \) and \( nb \to \infty. \)

### 8.3 Proof of Lemma 3

Let \( \delta_1 \in (0, 2\delta/3) \) such that \( t < \frac{2\delta-3\delta_1}{3(3+\delta)}. \) We set \( N = N_n = n^{-2/3}b^{-2-\delta}. \) For simplicity of notations, we suppress the dependence in \( n \) of our quantities. Then we set
\[ \xi_{g}^{(h,s,N)} = (-N) \lor \xi_{g}^{(h,s)} \land N - \mathbb{E} \left( (-N) \lor \xi_{g}^{(h,s)} \land N|\mathcal{T}_{g-1}^{(s)} \right). \]

We first show that
\[ \max_{h \in \mathcal{H}_n} \sum_{s=1}^{\ell} \sum_{g=1}^{k} \left| \xi_{g}^{(h,s)} - \xi_{g}^{(h,s,N)} \right| = o_{\mathbb{P}}(1). \]

To this end, we use the bound
\[
\left| \xi_{g}^{(h,s)} - \xi_{g}^{(h,s,N)} \right| \leq \left| \xi_{g}^{(h,s)} \right| I_{\xi_{g}^{(h,s)} > N} + \mathbb{E} \left[ \left| \xi_{g}^{(h,s)} \right| I_{\xi_{g}^{(h,s)} > N} | \mathcal{T}_{g-1}^{(s)} \right]
\]
\[
\leq \left| \xi_{g}^{(h,s)} \right| I_{\xi_{g}^{(h,s)} > N} + \frac{1}{N} \mathbb{E} \left[ \left| \xi_{g}^{(h,s)} \right|^{2} | \mathcal{T}_{g-1}^{(s)} \right].
\]

For \( \epsilon > 0 \), we have
\[
\mathbb{P} \left( \max_{h \in \mathcal{H}_n} \sum_{s=1}^{\ell} \sum_{g=1}^{k} \left| \xi_{g}^{(h,s)} \right| I_{\xi_{g}^{(h,s)} > N} > \epsilon \right)
\]
\[
\leq \mathbb{P} \left( \max_{h \in \mathcal{H}_n} \left| \xi_{g}^{(h,s)} \right| > N \right).
\]

Using the expression of \( N \) and the assumptions of the lemma, the latest probability tends to zero in probability. Moreover
\[
\frac{1}{N} \max_{h \in \mathcal{H}_n} \sum_{s=1}^{\ell} \sum_{g=1}^{k} \mathbb{E} \left( \left| \xi_{n,g}^{(h,s)} \right|^{2} | \mathcal{T}_{n,g-1}^{(s)} \right) = O_{\mathbb{P}} \left( \frac{1}{Nnb^{3}} \right) = o_{\mathbb{P}}(1).
\]
This show (27). To end the proof we have to show that

\[ \max_{h \in \mathcal{H}} \ell \sum_{s=1}^{k} \sum_{g=1}^{k} \xi_g^{(h,s,N)} = o_p(1). \]  

(28)

We will prove (28) using the exponential inequality of Freedman for martingales (see Freedman (1975)). Let \( \epsilon, \epsilon' > 0 \). First we choose \( M > 0 \) such that

\[ P \left( \max_{h \in \mathcal{H}} \ell \sum_{s=1}^{k} \sum_{g=1}^{k} \mathbb{E} \left( \left| \xi_g^{(h,s,N)} \right|^2 | \mathcal{T}_{n,g-1}^{(s)} \right) \right) > M \] 

Using the fact that

\[ \mathbb{E} \left( \left| \xi_g^{(h,s,N)} \right|^2 \right) \leq \mathbb{E} \left( \left| \xi_g^{(h,s)} \right|^2 \right), \]

we get

\[ P \left( \max_{h \in \mathcal{H}} \ell \sum_{s=1}^{k} \sum_{g=1}^{k} \xi_g^{(h,s,N)} \right) > \epsilon, \max_{h \in \mathcal{H}} \ell \sum_{s=1}^{k} \sum_{g=1}^{k} \mathbb{E} \left( \left| \xi_g^{(h,s)} \right|^2 \right) \leq \frac{M}{nb^3} \]

\[ \leq \sum_{h \in \mathcal{H}} \sum_{s=1}^{\ell} \sum_{g=1}^{k} \mathbb{P} \left( \left| \xi_g^{(h,s,N)} \right| > \epsilon \right) \leq \epsilon, \sum_{s=1}^{\ell} \sum_{g=1}^{k} \mathbb{E} \left( \left| \xi_g^{(h,s)} \right|^2 \right) \leq \frac{M}{nb^3} \]

\[ \leq |2\ell| \exp \left( \frac{-\epsilon^2}{2M\ell^2 + 4\epsilon \ell N} \right). \]

Then (28) follows from our conditions on \( b, \ell, N, \mathcal{H} \). The proof of Lemma 3 is now complete. □

8.4 Proof of Lemma 4

For the proof of Lemma 4, we need two additional lemmas which are stated below. For \( \theta \in \Theta_{0,\epsilon} \), the density of \( \xi_i(\theta) \) will be denoted by \( f_\theta \). Then, we have the expression

\[ f_\theta(w) = \mathbb{E} \left[ \frac{\sigma_j(\theta)}{\sigma_j(\theta_0)} f_\epsilon \left( \frac{\sigma_j(\theta) w + m_j(\theta) - m_j(\theta_0)}{\sigma_j(\theta_0)} \right) \right]. \]  

(29)

The following lemma is given without proof. The assertions given below are mainly a consequence of the Lebesgue theorem for the derivative of an integral depending on a parameter.

Lemma 5. Assume that assumptions A3 and A5 hold true.

1. We have \( \sup_{\theta \in \Theta_{0,\epsilon}, w \in \mathbb{R}} f_\theta(w) < \infty \). For \( \theta \in \Theta_{0,\epsilon} \), the function \( w \mapsto f_\theta(w) \) has a derivative \( f'_\theta \) such that \( \sup_{\theta \in \Theta_{0,\epsilon}, w \in \mathbb{R}} |f'_\theta(w)| < \infty \). Moreover, there exists a number \( C > 0 \) not depending on \( \theta, w \) such that

\[ |f'_\theta(w) - f'_{\theta_0}(w)| \leq C (1 + |w|) \| \theta - \theta_0 \|. \]
2. For each \( w \in \mathbb{R} \), the function \( \theta \mapsto f_\theta(w) \) is two times differentiable. Moreover, there exists a number \( C > 0 \) not depending on \( w, \theta \) such that

\[
|\hat{f}_\theta(w) - \hat{f}_{\theta_0}(w)| \leq C (1 + w^2) \|\theta - \theta_0\|, \quad |\hat{f}_\theta(w)| \leq C (1 + |w|)
\]

and

\[
|\hat{f}_{\theta_0}(w_1) - \hat{f}_{\theta_0}(w_2)| \leq C (1 + |w_1|) \cdot |w_1 - w_2|.
\]

3. There exist two constants \( C_1 \) and \( C_2 \) such that

\[
|f_\theta(w) - f_{\theta_0}(w)| \leq (C_1 + C_2|w|) \|\theta - \theta_0\|.
\]

4. If \( F : \mathbb{R} \to \mathbb{R} \) a function continuously differentiable and with a compact support. Then, for \( \theta \in B(\theta_0, \epsilon) \), we have

\[
E[F(\varepsilon_i(\theta))] = \int F(w)f_\theta(w)dw, \quad E[\varepsilon_i(\theta)F'(\varepsilon_i(\theta))] = \int F(w)\hat{f}_\theta(w)dw.
\]

The next lemma is also given without proof because it results from simple computations.

**Lemma 6.** Assume that assumption **A3** holds true. We set \( U_j(\theta) = \frac{m_j(\theta_0) - m_j(\theta)}{\sigma_j(\theta)} \) and \( V_j(\theta) = \frac{\sigma_j(\theta_0)}{\sigma_j(\theta)} \). Note that \( \varepsilon_j(\theta) = U_j(\theta) + V_j(\theta)\varepsilon_j \).

1. We have \( E \left| \frac{U_j^2}{\varepsilon_j} \right|_\infty < \infty, \quad E \left| \frac{U_jV_j}{\varepsilon_j^2} \right|_\infty < \infty, \quad E \left| \frac{V_j}{\varepsilon_j} \right|_\infty < \infty, \quad E \left| \varepsilon_j^2 \right|_\infty < \infty \) and \( E \left| \varepsilon_j \right|_\infty < \infty \).

2. We have \( E \left[ \sup_{v \in I} |L_{v,i}|^2  \right] < \infty, \quad E \left[ \sup_{v \in I} |\hat{L}_{v,i}|^2 \right] < \infty \) and \( E \left[ \sup_{v \in I} |\hat{L}_{v,i}|^2 \right] < \infty \).

3. Let \( \eta \) be a positive number. There exists a positive real number \( C \) such that

\[
\sup_{|v - v'| \leq \eta} |L_{v,i}(\theta) - L_{v,i}(\zeta)| \leq C \eta \left( 1 + \sup_{v \in I} |\hat{L}_{v,i}|^2 \right)
\]

and

\[
\sup_{|v - v'| \leq \eta} \left| \hat{L}_{v,i}(\theta) - \hat{L}_{v,i}(\zeta) \right| \leq C \eta \left( |\hat{\sigma}_i|_{\infty, \epsilon} + \sup_{v \in I} |\hat{L}_{v,i}|^2 \right).
\]

**Proof of Lemma 4.** We only prove the result for the conditionally heteroscedastic case, the homoscedastic uses similar arguments and is simpler.

1. It is only necessary to prove that

\[
\sum_{(i,j) \in \mathcal{I}_n} \sup_{\theta \in B(\theta_0, \epsilon)} E_{Y_{ij}} \|\hat{A}_{v,ij}(\theta)\|^2 = O_p \left( \frac{n^2}{\theta^3} \right).
\]
We recall that
\[ \hat{A}_{v,ij}(\theta) = \sigma_{\ell}^{-1}(\theta) K_b [L_{v,\ell t}(\theta) - \varepsilon_{j\ell}(\theta)] + \sigma_{\ell}^{-1}(\theta) \left[ \hat{L}_{v,\ell t}(\theta) - \hat{\varepsilon}_{j\ell}(\theta) \right] K'_b [L_{v,\ell t}(\theta) - \varepsilon_{j\ell}(\theta)]. \]

We define
\[ U_{j\ell}(\theta) = \frac{m_{j\ell}(\theta_0) - m_{j\ell}(\theta)}{\sigma_{j\ell}(\theta)}, \quad V_{j\ell}(\theta) = \frac{\sigma_{j\ell}(\theta_0)}{\sigma_{j\ell}(\theta)}. \]

Then \( \varepsilon_{j\ell}(\theta) = U_{j\ell}(\theta) + V_{j\ell}(\theta)\varepsilon_j \). Moreover
\[
\mathbb{E}_\varepsilon \left[ \| \hat{\varepsilon}_{j\ell}(\theta) \|^2 \cdot |K'_b (L_{v,\ell t}(\theta) - \varepsilon_{j\ell}(\theta))|^2 \right] \\
= \int \frac{1}{b^3} \left[ |\dot{U}_{j\ell}(\theta) + \frac{\dot{V}_{j\ell}(\theta)}{V_{j\ell}(\theta)} (L_{v,\ell t}(\theta) - U_{j\ell}(\theta) - bw) |^2 \cdot |K'(w)|^2 f_\varepsilon \left( \frac{L_{v,\ell t}(\theta) - U_{j\ell}(\theta) - bw}{V_{j\ell}(\theta)} \right) \right] dw \\
\leq C \frac{1}{b^3} \left[ 1 + \left| \frac{\dot{U}_{j\ell}}{\dot{V}_{j\ell}} \right|^2_{\infty,\varepsilon} + \left| \frac{\dot{V}_{j\ell}}{V_{j\ell}} \right|^2_{\infty,\varepsilon} + \sup_{v\in I} |L_{v,\ell t}|^2_{\infty,\varepsilon} \cdot \left| \frac{U_{j\ell} \dot{V}_{j\ell}}{V_{j\ell}} \right|^2_{\infty,\varepsilon} \right].
\]

Then we get
\[
\mathbb{E}_{V_{j\ell}} \left[ \| \hat{A}_{v,ij}(\theta) \|^2 \right] \\
\leq C \frac{1}{b^2} \left[ 1 + \left| \frac{\dot{m}_{j\ell}}{\dot{\sigma}_{j\ell}} \right|^2_{\infty,\varepsilon} + \sup_{v\in I} |L_{v,\ell t}|^2_{\infty,\varepsilon} + \mathbb{E} \left| \hat{U}_{j\ell} \right|^2_{\infty,\varepsilon} + \mathbb{E} \left| \frac{\dot{V}_{j\ell}}{V_{j\ell}} \right|^2_{\infty,\varepsilon} + \sup_{v\in I} |L_{v,\ell t}|^2_{\infty,\varepsilon} \cdot \mathbb{E} \left| \frac{U_{j\ell} \dot{V}_{j\ell}}{V_{j\ell}} \right|^2_{\infty,\varepsilon} \right].
\]

The result follows from assumption A3 and Lemma 6.

2. Since
\[ \hat{\varepsilon}_{j\ell}(\theta) = \frac{\dot{m}_{j\ell}(\theta)}{\sigma_{j\ell}(\theta)} - \frac{\dot{\sigma}_{j\ell}(\theta)}{\sigma_{j\ell}(\theta)} \varepsilon_{j\ell}(\theta), \]
we have, using the compact support of the kernel \( K \), and the equality \( \frac{\dot{\sigma}_{j\ell}}{\sigma_{j\ell}} = \frac{\dot{\sigma}_{j\ell}}{\sigma_{j\ell}}^2 \),
\[
\| \hat{\varepsilon}_{j\ell}(\theta) \| \cdot |K'_b (L_{v,\ell t}(\theta) - \varepsilon_{j\ell}(\theta))| \leq C \frac{1}{b^2} \left[ |\dot{m}_{j\ell}, \sigma_{j\ell}|_{\infty,\varepsilon} + |\dot{\sigma}_{j\ell}, \sigma_{j\ell}|_{\infty,\varepsilon} \right] \cdot \left[ 1 + \sup_{v\in I} |L_{v,\ell t}|_{\infty,\varepsilon} \right].
\]

Then we conclude that
\[
\sup_{v\in I} \| \hat{A}_{v,ij} \| \\
\leq C \frac{1}{b^2} \left[ |\dot{\sigma}_{j\ell}, \sigma_{j\ell}|_{\infty,\varepsilon} + \sup_{v\in I} |L_{v,\ell t}|_{\infty,\varepsilon} + |\dot{m}_{j\ell}, \sigma_{j\ell}|_{\infty,\varepsilon} + |\dot{\sigma}_{j\ell}, \sigma_{j\ell}|_{\infty,\varepsilon} \right] \cdot \left[ 1 + \sup_{v\in I} |L_{v,\ell t}|_{\infty,\varepsilon} \right].
\]

From the assumption A3, we have \( \max_{1 \leq j \leq n} |\dot{m}_{j\ell}, \sigma_{j\ell}|_{\infty,\varepsilon} = O_\mathbb{P} (n^{1/3}) \) and \( \max_{1 \leq j \leq n} |\dot{\sigma}_{j\ell}, \sigma_{j\ell}|_{\infty,\varepsilon} = O_\mathbb{P} (n^{1/3}) \). Then the result follows from the point 2 of Lemma 6.
3. If \((v, w) \in I^2\) and \((\theta, \zeta) \in \Theta_{0, \epsilon}\) are such that \(|v - w| \leq \eta\) and \(\|\theta - \zeta\| \leq \eta\), some basic computations lead to the inequality

\[
\left| \dot{A}_{v,ij}(\theta) - \dot{A}_{w,ij}(\zeta) \right| \leq \frac{C\eta}{b^3} \left[ 1 + \left| \sigma^2_{i\ell, \sigma^2_{i\ell}} \right|_{\infty, \epsilon} + \left| \sigma^2_{i\ell, \sigma^2_{i\ell}} \right|_{\infty, \epsilon}^2 \right] + \frac{C\eta}{b^3} \left[ \sup_{v \in I} \left| \dot{L}_{v,\ell} \right|_{\infty, \epsilon}^2 + \left| \dot{\varepsilon}_{j\ell} \right|_{\infty, \epsilon} + \sup_{v \in I} \left| L_{v,\ell} \right|_{\infty, \epsilon} + \left| \dot{\varepsilon}_{j\ell} \right|_{\infty, \epsilon} \right].
\]

Then the result follows from assumption \(A3\) and Lemma 6. □

9 Proofs of assertions (18) and (19) in the proof of Theorem 2

9.1 Proof of assertion (18)

The following notations will be needed. We define

\[
\Gamma^{(1)}_i(v, \theta) = \frac{\sigma^{-1}_{i\ell}(\theta)}{\dot{\sigma}_{i\ell}(\theta)} \int K(w) f_\theta (L_{v,\ell}(\theta) - bw) \, dw,
\]

\[
\Gamma^{(2)}_i(v, \theta) = \frac{\dot{L}_{v,\ell}(\theta)}{\dot{\sigma}_{i\ell}(\theta)} \int K(w) f'_\theta (L_{v,\ell}(\theta) - bw) \, dw,
\]

\[
\Gamma^{(3)}_i(v, \theta) = \frac{1}{\sigma_{i\ell}(\theta)} \int K(w) f_\theta (L_{v,\ell}(\theta) - bw) \, dw.
\]

Note that from Lemma 5, 4., we have

\[
\mathbb{E}_{Y_{i\ell} A_{v,ij}(\theta)} = \sum_{h=1}^{3} \Gamma^{(h)}_i(v, \theta).
\]

Then assertion (21) will follow if we show that for \(h = 1, 2, 3\),

\[
\frac{1}{n^2} \sum_{(i,j) \in I^2} \sup_{v \in I, \theta \in \Theta_{0,n}} \left\| \Gamma^{(h)}_i(v, \theta) - \Gamma^{(h)}_i(v, \theta_0) \right\| = o_P(1). \tag{30}
\]

The proof of (30) follows from the following bounds, Assumption \(A3\), Lemma 5 and Lemma 6.

- For \(h = 1\), we have

\[
\sup_{v \in I, \theta \in \Theta_{0,n}} \left\| \Gamma^{(1)}_i(v, \theta) - \Gamma^{(1)}_i(v, \theta_0) \right\| \leq \frac{C}{\sqrt{n}} \left[ \left| \sigma^{-1}_{i\ell} \right|_{\infty, \epsilon} + \left| \sigma^{-1}_{i\ell} \right|_{\infty, \epsilon} \left( \sup_{v \in I} \left| \dot{L}_{v,\ell} \right|_{\infty, \epsilon} + C_1 + C_2 \left( \sup_{v \in I} \left| L_{v,\ell} \right|_{\infty, \epsilon} + 1 \right) \right) \right],
\]

where \(C_1\) and \(C_2\) are the constants given in Lemma 5 (3).
• For $h = 2$, we have

$$
\sup_{v \in I, \theta \in \Theta_{0,n}} \left\| \Gamma_{i}^{(2)}(v, \theta) - \Gamma_{i}^{(2)}(v, \theta_0) \right\| \leq \frac{C}{\sqrt{n}} \left[ \sup_{v \in I} \left| \tilde{L}_v,i|t \right|_{\infty, \epsilon} + \sup_{v \in I} \left| \bar{L}_v,i|t \right|_{\infty, \epsilon} \cdot |\tilde{\sigma}_{it}, \sigma_{it}|_{\infty, \epsilon} \right] \\
+ \frac{C}{\sqrt{n}} \sup_{v \in I} \left| \dot{L}_v,i|t \right|_{\infty, \epsilon} \cdot \left( 1 + \sup_{v \in I} \left| L_v,i|t \right|_{\infty, \epsilon} + \sup_{v \in I} \left| \dot{L}_v,i|t \right|_{\infty, \epsilon} \right).
$$

• Finally we have

$$
\sup_{v \in I, \theta \in \Theta_{0,n}} \left\| \Gamma_{i}^{(3)}(v, \theta) - \Gamma_{i}^{(3)}(v, \theta_0) \right\| \leq \frac{C}{\sqrt{n}} \left[ |\tilde{\sigma}_{it}, \sigma_{it}|_{\infty, \epsilon} \left( 1 + \sup_{v \in I} \left| L_v,i|t \right|_{\infty, \epsilon} \right) \right] \\
+ \frac{C}{\sqrt{n}} \left[ 1 + \sup_{v \in I} \left| L_v,i|t \right|_{\infty, \epsilon} + \sup_{v \in I} \left| \dot{L}_v,i|t \right|_{\infty, \epsilon} \cdot \left( 1 + \sup_{v \in I} \left| L_v,i|t \right|_{\infty, \epsilon} \right) \right].
$$

Using the integrability properties stated in Lemma 6 and assumption A3, assertion (18) given in the paper follows.

### 9.1.1 Proof of assertion (19)

We set

$$
\Delta_{v,i}^{(1)} = \frac{-\tilde{\sigma}_{it}(\theta_0)}{\sigma_{it}^2(\theta_0)} f_{\theta_0} \left( L_v,i|t(\theta_0) \right), \quad \Delta_{v,i}^{(2)} = \frac{\tilde{L}_v,i|t(\theta_0)}{\sigma_{it}(\theta_0)} f_{\theta_0}' \left( L_v,i|t(\theta_0) \right), \quad \Delta_{v,i}^{(3)} = \frac{1}{\sigma_{it}(\theta_0)} f_{\theta_0}' \left( L_v,i|t(\theta_0) \right).
$$

Using Lemma 5 and Lemma 6, it is easily seen that for $h = 1, 2, 3$,

$$
\frac{1}{n^2} \sum_{(i,j) \in \mathcal{I}_n} \sup_{v \in I} \left\| \Gamma_{v,i}(\theta_0) - \Delta_{v,i} \right\| = o_p(1).
$$

To end the proof of assertion (19), it remains to show that for $h = 1, 2, 3$,

$$
\sup_{v \in I} \frac{1}{n^2} \left| \sum_{(i,j) \in \mathcal{I}_n} \left[ \Delta_{v,i}^{(h)} - \mathbb{E} \left( \Delta_{v,i}^{(h)} \right) \right] \right| = o_p(1). \tag{31}
$$

Note that $\sum_{h=1}^{3} \mathbb{E} \left( \Delta_{v,i}^{(h)} \right) = \hat{h}_{\theta_0}(v)$. To show (31), we first notice that $\sup_{n \in \mathbb{N}^*, v \in I} \mathbb{E} \left\| \Delta_{v,i}^{(h)} \right\| < \infty$ for $h = 1, 2, 3$. Then, since $\ell = o(n)$, it is easily seen that

$$
\sup_{v \in I} \left| \frac{1}{n^2} \sum_{(i,j) \in \mathcal{I}_n} \left[ \Delta_{v,i}^{(h)} - \mathbb{E} \left( \Delta_{v,i}^{(h)} \right) \right] - \frac{1}{n} \sum_{i=1}^{n'} \left[ \Delta_{v,i}^{(h)} - \mathbb{E} \left( \Delta_{v,i}^{(h)} \right) \right] \right| = o_p(1).
$$

Now, we set for $v \in I$ and $h = 1, 2, 3$,

$$
G_{n}^{(h)}(v) = \frac{1}{n} \sum_{i=1}^{n'} \left[ \Delta_{v,i}^{(h)} - \mathbb{E} \left( \Delta_{v,i}^{(h)} \right) \right].
$$
Then assertion (19) will follow if we show that
\[
\sup_{v \in I} \left\| G_n^{(h)} (v) \right\| = o_P(1), \quad h = 1, 2, 3.
\] (32)

We have using the \( \ell \)-dependence,
\[
\mathbb{E} \left| G_n^{(h)} (v) \right|^2 \leq \frac{\ell}{n^2} \sum_{s=1}^{\ell} \sum_{y=0}^{k-1} \text{Var} \left( M_{s+y \ell}^{(h)} (v) \right).
\]

Moreover, using the fact that \( \ell = o(n) \) and that \( \sup_{n, i \geq 1} \sup_{v \in I} \text{Var} \left( M_i^{(h)} (v) \right) \) is bounded, we get \( G_n^{(h)} (v) = o_P(1) \) for each \( v \in I \). Now, (32) will follow if we show that \( \sup_{v \neq 0} \frac{\| G_n^{(h)} (v) - G_n^{(h)} (\bar{v}) \|}{\| v - \bar{v} \|} = O_P(1) \). But this is a consequence of the following bounds. First, there exists \( C > 0 \) such that
\[
\left\| \Delta_{v,i}^{(1)} - \Delta_{v',i}^{(1)} \right\| \leq C \left| \hat{\sigma}_i t, \sigma_i t \right|_{\infty, \ell} |v - v'|,
\]
\[
\left\| \Delta_{v,i}^{(2)} - \Delta_{v',i}^{(2)} \right\| \leq C \left[ \hat{\sigma}_i t, \sigma_i t \right|_{\infty, \ell} + \sup_{v \in I} \left| \hat{L}_{v,i} t \right|_{\infty, \ell} \right] \cdot |v - v'|.
\]

For \( h = 3 \), we set \( E_1 = \mathbb{E} \left[ \hat{\sigma}_i t (\theta_0) \right] \) and \( E_2 = \mathbb{E} \left[ \hat{m}_i t (\theta_0) \right] \). Then \( \dot{f}_{\theta_0} (w) = E_1 (f_{\dot{\xi}} (w) + w f_{\dot{\xi}}' (w)) + E_2 f_{\dot{\xi}}' (w) \). Then, using assumption A5, we have
\[
\left\| \Delta_{v,i}^{(3)} - \Delta_{v',i}^{(3)} \right\| \leq C \left[ |v - v'| + \sup_{v \in I} \left| \hat{L}_{v,i} t \right|_{\infty, \ell} \right].
\]

In the previous bounds, the real number \( C \) does not depend on \( v, \bar{v} \in I \). Then (32) follows and the proof of assertion (19) is now complete.

10 Checking the regularity assumptions on densities

10.1 Density regularities of ARCH processes

Here, we assume that \( (X_t) \) is a stationary ARCH process defined by
\[
X_t = \varepsilon_t \sigma_t, \quad \sigma_t^2 = \alpha_0 + \sum_{j \geq 1} \alpha_j X_{t-j}^2.
\]

We assume here that \( \sum_{j=1}^{\infty} \alpha_j < \infty \) and that \( f_{\dot{\xi}} \) is bounded. We set \( \mu(dx) = \sup_{|z| \leq \alpha_0} f_{\dot{\xi}} (x + z) dx \).

We denote by \( f_{\sigma^2} \) the probability density of the conditional variance \( \sigma_t^2 \) and for \( v \neq 0 \), \( g_v (x) = \frac{2v}{\pi} f_{\sigma^2} \left( \frac{x^2}{\sigma_t^2} \right) \) which is well defined for \( x \neq 0 \). We also set \( G(x) = \sup_{v \in I} g_v (x) \) and for a compact interval \( I \) which does not contain 0, \( G_I = \{ g_v : v \in I \} \).

**Lemma 7.** 1. Assume that \( \alpha_1, \alpha_2 > 0 \). Then \( f_{\sigma^2} \) is bounded. Moreover, if \( \mathbb{E} \sigma_t < \infty \), then for all \( v \neq 0 \), we have \( \int g_v (x)^2 \mu(dx) < \infty \).
2. Assume that $\alpha_1, \alpha_2, \alpha_3 > 0$. Then there exists a constant $C > 0$ such that for $s_1, s_2 \geq 0$, we have
\[
|f_{\sigma^2}(s_2) - f_{\sigma^2}(s_1)| \leq C \left(1 + \sqrt{|s_1|} \wedge \sqrt{|s_2|}\right) \cdot \sqrt{|s_2 - s_1|}.
\]

3. In addition to the previous point, assume that there exists $\delta \in (0, \frac{1}{2})$ such that $\mathbb{E} \sigma_t^{3+\delta} < \infty$ and that $x \mapsto |x|^{\frac{3}{2}+\delta} f_\varepsilon(x)$ is bounded. Then there exists a real number $\alpha > 0$ such that $\int G(x)^{2+\alpha} \mu(dx) < \infty$. Moreover there exists some constants $\zeta, C > 0$ such that
\[
N_{\| \cdot \|}(\varepsilon, G_1, L^2(\mu)) \leq C e^{-\zeta}.
\]

**Proof of Lemma 7.** Before proving the lemma, we first derive an expression for $f_{\sigma^2}$ involving conditional distributions. We will use for $j \geq 1$ the notation $z_j = (z_j, z_{j+1}, \ldots)$, we set $k_3(z_3) = \alpha_0 + \sum_{j=3}^{\infty} \alpha_j z_j^2$ and
\[
s(z_1) = \sqrt{\alpha_0 + \sum_{j=1}^{\infty} \alpha_j z_j^2}.
\]
The measure $\nu$ will denote the probability distribution of $(X_{t-1}, X_{t-2}, \ldots)$. Moreover, we set
\[
r(z_1, z_2|z_3) = \frac{1}{s(z_2)} f_\varepsilon \left(\frac{z_1}{s(z_2)}\right) \frac{1}{s(z_3)} f_\varepsilon \left(\frac{z_2}{s(z_3)}\right)
\]
and
\[
\tilde{r}(z_1, z_2|z_3) = \frac{1}{\sqrt{\alpha_1 \cdot \alpha_2}} r \left(\frac{z_1}{\sqrt{\alpha_1}}, \frac{z_2}{\sqrt{\alpha_2}}|z_3\right).
\]
If $h : \mathbb{R} \to \mathbb{R}$ is a bounded and measurable function, we have
\[
\mathbb{E} h \left(\sigma_t^2\right) = \int \int \int h \left(\alpha_1 z_1^2 + \alpha_2 z_2^2 + k_3(z_3)\right) r(z_1, z_2|z_3) dz_1 dz_2 d\nu(z_3)
\]
\[
= \int \int \int h \left(z_1^2 + z_2^2 + k_3(z_3)\right) \tilde{r}(z_1, z_2|z_3) dz_1 dz_2 d\nu(z_3)
\]
\[
= \int_0^\infty \int_{-\pi}^{\pi} \int h \left(\rho^2 + k_3(z_3)\right) \tilde{r}(\rho \cos(\phi), \rho \sin(\phi)|z_3) \rho d\rho d\phi d\nu(z_3)
\]
\[
= \int \int_{k_3(z_3) \leq s} \int_{-\pi}^{\pi} h(s) \frac{1}{2} \tilde{r}(\sqrt{s - k_3(z_3)} \cos(\phi), \sqrt{s - k_3(z_3)} \sin(\phi)|z_3) ds d\phi d\nu(z_3).
\]
Then we deduce that for $s \geq 0$,
\[
f_{\sigma^2}(s) = \frac{1}{2} \int_{k_3(z_3) \leq s} \int_{-\pi}^{\pi} \tilde{r}(\sqrt{s - k_3(z_3)} \cos(\phi), \sqrt{s - k_3(z_3)} \sin(\phi)|z_3) d\phi d\nu(z_3). \tag{33}
\]
1. The fact that $f_{\sigma^2}$ is bounded is a consequence of the expression (33), using the fact that $f_\varepsilon$ and then $\tilde{r}$ are bounded. Moreover, we have
\[
\int g(x)^2 \mu(dx) \leq 4 v^2 \cdot \|f_\varepsilon\|_{\infty} \int \frac{1}{x^4} f_{\sigma^2} \left(\frac{v^2}{x^2}\right) dx
\]
\[
\leq \frac{4}{|v|} \|f_\varepsilon\|_{\infty} \int_0^\infty \sqrt{y} f_{\sigma^2}(y) dy.
\]
This bound gives the result.
2. We use the expression (33). Using some basic computations, it is easily seen that for real numbers \( z_1, z_2 \),
\[
|r(z_1, z_2| z_3) - r(\bar{z}_1, \bar{z}_2| z_3)| \leq C \left( |z_1 - \bar{z}_1| + (1 + |z_1| \wedge |\bar{z}_1|) \cdot |z_2 - \bar{z}_2| \right)
\]
for some constant \( C > 0 \). Setting now for \( s \geq k_3(z_3) \),
\[
h(s, \phi, z_3) = \bar{r}\left[ \sqrt{s - k_3(z_3)} \cos(\phi), \sqrt{s - k_3(z_3)} \sin(\phi) \right| z_3],
\]
we have for \( s_2 \geq s_1 \geq k_3(z_3) \),
\[
|h(s_1, \phi, z_3) - h(s_2, \phi, z_3)| \leq C \left( 1 + \sqrt{s_1 \wedge s_2} \right) \cdot \sqrt{|s_2 - s_1|}.
\]
We deduce that for \( s_1, s_2 \geq 0 \),
\[
|f_{\sigma^2}(s_2) - f_{\sigma^2}(s_1)| \leq C \left( 1 + \sqrt{s_1 \wedge s_2} \right) \cdot \sqrt{|s_2 - s_1|} + \mathbb{P} \left( s_1 < k_3(X_{t-3}, \ldots) \leq s_2 \right).
\]
Moreover, it is easily seen that \( k_3(X_{t-3}, \ldots) \) has a density \( q_\lambda \) such that
\[
q_\lambda(x) \leq C \int_{k_4(z_4) \leq x} (x - k_4(z_4))^{-1/2} \, d\nu(z_4),
\]
where \( k_4(z_4) = \alpha_0 + \sum_{j \geq 4} \alpha_j z_j^4 \). Then, it can be shown that
\[
\mathbb{P} \left( s_1 < k_3(X_{t-3}, \ldots) \leq s_2 \right) \leq C \sqrt{|s_2 - s_1|}.
\]
This proves the second point of this lemma.

3. We first show that under our assumptions,
\[
\sup_{x > 0} x^\frac{3}{4} + \frac{4}{7} f_{\sigma^2}(x) < \infty. \tag{34}
\]
We first observe that
\[
|u|^{\frac{3}{2} + \delta} \bar{r} \left( u \cos \phi, u \sin \phi \right| z_3) \leq C \left[ |u \cos \phi|^{\frac{3}{2} + \delta} + |u \sin \phi|^{\frac{3}{2} + \delta} \right] \cdot \bar{r} \left( u \cos \phi, u \sin \phi \right| z_3)
\]
\[
\leq C \sup_{y > 0} \left\{ y^{\frac{3}{2} + \delta} f_\varepsilon(y) \right\} \cdot \left[ \frac{\alpha_0 + \alpha_1 u^2 \sin^2 \phi + \sum_{j \geq 2} \alpha_j z_j^2}{s(z_3)^{\frac{1}{2} + \delta}} \right] f_\varepsilon \left( \frac{u \sin \phi}{\sqrt{\alpha_2 s(z_3)}} \right)
\]
\[
+ C \sup_{y > 0} \left\{ y^{\frac{3}{2} + \delta} f_\varepsilon(y) \right\} \cdot s(z_3)^{\frac{1}{2} + \delta}
\]
\[
\leq C \sup_{y > 0} \left\{ y^{\frac{3}{2} + \delta} f_\varepsilon(y) \right\} \cdot \left[ \sup_{y > 0} \left\{ y^{\frac{3}{2} + \delta} f_\varepsilon(y) \right\} \right] + \alpha_0 + \sum_{j \geq 2} \alpha_j z_j^2 + s(z_3)^{\frac{1}{2} + \delta}
\]

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Then we deduce that the function \( \vartheta \) defined by
\[
\vartheta(z_3) = \sup_{u \in \mathbb{R}, \phi \in (-\pi, \pi)} |u|^{2+\delta} E_{\phi} (u \cos \phi, u \sin \phi | z_3)
\]
satisfies \( \mathbb{E} [\vartheta (X_{t-3}, X_{t-4}, \ldots)] < \infty \). Using the expression (33), condition \( \mathbb{E} \sigma^2_t < \infty \) and the decomposition \( x = x - k_3(z_3) + k_3(z_3) \), we get the bound
\[
x^{2+\delta} f_{12}(x) \\
\leq C \left( \mathbb{E} \left[k_3(X_{t-3}, X_{t-4}, \ldots) x^{2+\delta} \right] + \mathbb{E} [\vartheta (X_{t-3}, X_{t-4}, \ldots)] \right) \\
< \infty.
\]
This shows (34). For simplicity, we now assume that \( I \subset (0, \infty) \) (the case \( I \subset (-\infty, 0) \) is identical). First, we show that there exists \( o > 0 \) such that \( \int G(x)^{2+o} \mu(dx) < \infty \). We have Since \( I \) is compact and \( \sigma^2_t(\theta_0) \) is bounded from below, there exists \( \varpi > 0 \) such that \( G(x) = 0 \) when \( x > \varpi \). We choose \( o \) such that \( \frac{3}{4} + \frac{\delta}{2} = 1 - \frac{1-o}{2(2+o)} \). We get
\[
\int_0^\infty G(x)^{2+o} dx \leq C \int_0^\infty \frac{1}{x^{1-o}} \left| x^{2+\delta + \frac{o}{2}} \right| \sup_{v \in I} f_{\sigma^2} \left( \frac{v^2}{x^2} \right) dx \\
\leq C \left[ \sup_{y>0} \left\{ y^{\frac{3}{4}+\frac{\delta}{2}} f_{\sigma^2}(y) \right\} \right]^{2+o}.
\]
This shows that \( \int_0^\infty G(x)^{2+o} dx < \infty \). Finally, we consider the bracketing numbers of the family \( G_I \). Let \( \eta \in (0, 1) \) be such that \( 8\eta + (\frac{1}{2} - \delta)(2-2\eta) < 1 \). Let \( v_1, v_2 \in I \). We have
\[
|g_{v_1}(x) - g_{v_2}(x)| \leq CG(x)|v_1 - v_2| + \frac{2v}{x^2} \left| f_{\sigma^2} \left( \frac{v_1^2}{x^2} \right) - f_{\sigma^2} \left( \frac{v_2^2}{x^2} \right) \right| \\
\leq C \left[ G(x)|v_1 - v_2| + \frac{G(x)^{1-\eta}}{x^{2\eta}} \left| f_{\sigma^2} \left( \frac{v_1^2}{x^2} \right) - f_{\sigma^2} \left( \frac{v_2^2}{x^2} \right) \right| \right]^{\eta} \\
\leq C \left[ G(x) + G(x)^{1-\eta} \left( |v_1 - v_2|^{3\eta} + |v_1 - v_2|^{4\eta} \right) \right] \cdot |v_1 - v_2|^{\eta}.
\]
Since we have
\[
\int_0^\infty G(x)^{2+o} dx < \infty, \quad \int_0^\infty G(x)^{2-2\eta} x^{-8\eta} dx < \infty,
\]
where the second integrability condition follows from the assumption on \( \eta \) and (34), the bound given for \( N^\parallel (\epsilon, G_I, L^2(\mu)) \) easily follows (see van der Vaart (1998), Example 19.7, for \( \eta = 1 \) and \( \mu \) a probability measure, but the arguments are similar in our case). This completes the proof of the lemma. \( \square \)

### 10.2 A result for ARMA-GARCH processes

**Lemma 8.** Assume that \((\psi_j)_{j \geq 1}\) and \((\alpha_j)_{j \geq 0}\) are two summable sequences of real numbers such that \( \alpha_i \geq 0 \) for \( i \geq 0 \) and \( \alpha_i > 0 \) for \( 0 \leq i \leq 3 \) and there exists an integer \( q \geq 1 \) such that \( \psi_q \neq 0 \). Let \((Z_t)_{t \in \mathbb{Z}}\) be a stationary process of real random variables such that \( \mathbb{E} Z^2_t < \infty \) and such that the conditional distribution of \( Z_t | Z_{t-1}, Z_{t-2}, \ldots \) has a bounded density. Then the density \( \omega \) of the couple \((\sum_{j=1}^\infty \psi_j Z_{t-j}, \sqrt{\alpha_0 + \sum_{j=1}^\infty \alpha_j Z^2_{t-j}})\) satisfies \( \omega(x, y) \leq Cy \) for a positive constant \( C \).
Proof of Lemma 8. We set $z_i = (z_i, z_{i+1}, \ldots)$ and we denote by $\tilde{f}(\cdot | z_1)$ the conditional density of $Z_t$ given that $Z_{t-i} = z_i$ for $i \geq 1$. We consider two cases.

1. We first assume that $q \leq 3$. We set
$$k_1(z_4) = \sum_{j \geq 4} \psi_j z_j, \quad k_2(z_4) = \alpha_0 + \sum_{j \geq 4} \alpha_j z_j^2$$
and $r = ||a||$ with $a = \left(\frac{\psi_i}{\sqrt{\alpha_i}}\right)_{1 \leq i \leq 3}$. We also set
$$\zeta(z_1) = \frac{1}{\sqrt{\alpha_1 \alpha_2 \alpha_3}} \tilde{f} \left(\frac{z_1}{\sqrt{\alpha_1}}, \frac{z_2}{\sqrt{\alpha_2}}, \frac{z_3}{\sqrt{\alpha_3}} ; z_4\right) \cdot \tilde{f} \left(\frac{z_2}{\sqrt{\alpha_2}}, \frac{z_3}{\sqrt{\alpha_3}} ; z_3\right) \cdot \tilde{f} \left(\frac{z_3}{\sqrt{\alpha_3}} ; z_4\right).$$
We also denote by $R$ the rotation of $\mathbb{R}^3$ such that $Re_1 = a/r$ where $e_1 = (1,0,0)$. For simplicity of notations, we only use one sign “integral” and do not precise the boundaries for integration in the next computations. Then if $\nu$ denotes the probability distribution of $(Z_{t-4}, Z_{t-5}, \ldots)$, we have
$$\mathbb{E} h \left(\sum_{j=1}^{\infty} \psi_j z_{t-j}, \sqrt{\alpha_0 + \sum_{j=1}^{\infty} \alpha_j z_{t-j}^2}\right)$$
$$= \int h \left(\sum_{j=1}^{3} a_j z_j + k_1(z_4), \sqrt{\sum_{j=1}^{3} z_j^2 + k_2(z_4)}\right) \zeta(z_1) dz_1 dz_2 dz_3 d\nu(z_4)$$
$$= \int h \left(r z_1 + k_1(z_4), \sqrt{\sum_{j=1}^{3} z_j^2 + k_2(z_4)}\right) \zeta \left(\mathcal{R}(z_1, z_2, z_3, z_4)\right) dz_1 dz_2 dz_3 d\nu(z_4)$$
$$= \int h(x,y) \zeta \left(\mathcal{R}\left(\frac{x - k_1(z_4)}{\alpha}\right), \sqrt{y^2 - \left[\frac{x - k_1(z_4)}{\alpha}\right]^2} - k_2(z_4) \cos \phi, \sqrt{y^2 - \left[\frac{x - k_1(z_4)}{\alpha}\right]^2} - k_2(z_4) \sin \phi\right), z_4\right)$$
$$\frac{y}{r} dx dy d\phi d\nu(z_4).$$

Since $\zeta$ is bounded, it is easily seen that $\omega(x,y) \leq Cy$ for a positive constant $C$.

2. We next consider the case $q \geq 4$. We set
$$\zeta_1(z_1) = \prod_{i=1}^{2} \tilde{f}(z_i | z_{i+1}), \quad \zeta_2(z_3) = \prod_{i=3}^{q-1} \tilde{f}(z_i | z_{i+1})$$
and $k_3(z_3) = \alpha_0 + \sum_{j=3}^{\infty} \alpha_j z_j^2$. For simplicity of notations, we assume that $\alpha_1 = \alpha_2 = 1$ (otherwise, as in the previous point, a change of variables is needed in the computations.
Since the noise density \( f \) paper for the bandwidth and the kernel are automatically satisfied here. We then check (\( C_1 \)). We first apply Theorem 2.11 Proof of Corollary 2.11 Adequation tests

Then we get

\[
\omega(x, y) = y \int \zeta_1 \left( \sqrt{y^2 - k_3(z_3)} \cos \phi, \sqrt{y^2 - k_3(z_3)} \cos \phi, \frac{x - \sum_{j \geq q+1} y^j z_j}{\psi_q}, z_{q+1} \right) \\
\zeta_2 \left( z_3, \ldots, z_{q-1}, \frac{x - \sum_{j \geq q+1} y^j z_j}{\psi_q}, z_{q+1} \right) \\
\tilde{f} \left( \frac{x - \sum_{j \geq q+1} y^j z_j}{\psi_q} \right) d\rho d\phi dz_3 \cdots dz_{q-1} d\nu(z_{q+1}).
\]

We deduce that there exists \( C > 0 \) such that \( \omega(x, y) \leq Cy \). \( \square \)

11 Adequation tests

11.1 Proof of Corollary 2

We first apply Theorem 2.2 in Liu and Wu (2010). The assumptions (C1) and (C4) used in that paper for the bandwidth and the kernel are automatically satisfied here. We then check (C3)'. Since the noise density \( f_\varepsilon \) is positive, the marginal density \( f_X \) is also positive. The conditional density of \( X_t|X_{t-1}, X_{t-2}... \) is given by \( \nu \mapsto \frac{1}{\sigma_t(\theta_0)} f_\varepsilon \left( \frac{v - m_t(\theta_0)}{\sigma_t(\theta_0)} \right) \). Using the condition

\[
\sup_{x \in \mathbb{R}} \left[ |f_\varepsilon(x)| + |f'_\varepsilon(x)| + |f''_\varepsilon(x)| \right] < \infty,
\]

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we easily deduce the boundedness of the conditional distribution and its two first derivatives and then \((C3)\). Finally, \((C2)\) is a straightforward consequence of our assumption \(A2\). Indeed, there exists a measurable function \(H : \mathbb{R}^n \to \mathbb{R}\) such that \(X_t = H(\varepsilon_t, \varepsilon_{t-1}, \ldots)\). Moreover, from Assumption \(A2\), we have, setting \(X_{t\ell} = H(\varepsilon_t, \ldots, \varepsilon_{t-\ell+1}, \varepsilon_{t-\ell}, \varepsilon'_{t-\ell-1}, \ldots)\),

\[
\mathbb{E}|X_t - X_{t\ell}|^\delta \leq Ca^\ell,
\]

for a suitable constant \(C > 0\). The dependence coefficient \(\theta_{t,s}\) defined in Liu and Wu (2010), Section 2, can be bounded by

\[
\theta_{t,s}^\delta \leq C\left[a^\ell + a^{\ell+1}\right].
\]

Then Assumption \((C2)\) follow. Applying Theorem 2.2 in Liu and Wu (2010), we deduce that the convergence given in Corollary 2 holds but with \(\Delta_n\) replaced by

\[
\Delta_n = \sqrt{n}b \sup_{v \in I} \frac{f_X^\#(v) - \mathbb{E}f_X^\#(v)}{f_X(v) \int K^2(u)du}.
\]

But since \(f_X\) has a bounded second derivative and \(nb^4 \to 0\), we have \(\sup_{v \in I} \mathbb{E}f_X^\#(v) = o(n^{-1/8})\). Moreover, from Theorem 1 (2.), we have \(\sqrt{n}b \sup_{v \in I} \left|\hat{f}_X(v) - f_X(v)\right| = O_P(n^{-1/2})\). We then deduce that

\[
\Delta_n - \Delta_n = O_P\left(\sqrt{b} + n^{-1/8}\right).
\]

Hence the same limit in distribution also holds for \(\Delta_n\) which proves the result. □

11.2 Simulation study

We investigate the performances of our test for a specific example. For homoscedastic data, our test is very similar to that of Kim et al. (2015) and we prefer to give a numerical comparison for ARCH processes. In this section, we consider the problem of testing \(H_0\): \(X\) follows an ARCH(3). Under \(H_0\), the dynamic of the process is given by

\[
X_t = 0.1 + \varepsilon_t \sqrt{0.1 + 0.3(X_{t-1} - 0.1)^2 + 0.3(X_{t-2} - 0.1)^2 + 0.3(X_{t-3} - 0.1)^2},
\]

and \(\varepsilon_t\) follows a standard Gaussian distribution. For this example, we have shown that the assumptions \(A1-A8\) are satisfied for any compact interval \(I\) which does not contain 0.1. We choose \(I = \{-1, 0\} \cup [0.2, 1]\). Our goal here is to see if our test can detect some departures from the null hypothesis in particular ARCH dynamics:

\[
X_t = 0.1 + \varepsilon_t \left(0.1 + 0.3|X_{t-1} - 0.1|^\delta + 0.3|X_{t-2} - 0.1|^\delta + 0.3|X_{t-3} - 0.1|^\delta\right)^{1/\delta}.
\]

Under \(H_1\), we assume that \(\delta = 1\). In the latter case, the volatility is linear w.r.t. absolute past returns. \(\delta = 2\) corresponds to the standard ARCH process. Deciding which power of the log returns has to be included in the volatility has been one of the important question in finance and led Ding et al. (1993) to consider power ARCH models as alternative to the standard ARCH processes. As pointed out in Kim et al. (2015), the type of convergence given in our Corollary 2 is quite slow and
for their statistics, the authors fix the critical values for the test by simulating many times a pivotal
statistics under a particular scenario for which the asymptotic behavior of the same statistics is still
valid. In our conditionally heteroscedastic case, we found that the critical values obtained by this
Monte-Carlo method can be quite sensitive to the simulation setup and we prefer to use the block
bootstrap method discussed in Künsch (1989). Suppose that a bootstrap sample \( X_1^*, X_2^*, \ldots, X_n^* \)
is available. We then replace the statistics \( \Delta_n \) by

\[
\Delta_n^* = \sqrt{n b} \sup_{v \in \mathcal{I}} \frac{|f_X^*(v) - \hat{f}_X(v) - f_X^*(v) + \hat{f}_X(v)|}{\sqrt{\hat{f}_X(v) \int K^2(u) du}},
\]

where \( f_X^* \) and \( \hat{f}_X \) are the versions of \( f_X^* \) and \( \hat{f}_X \) computed with the bootstrap sample. In the
i.i.d. case and when the density is known under \( H_0 \), the asymptotic validity of such bootstrap is
discussed in Bickel and Ren (2001) (see Example 7). We estimate the parameters (location and
ARCH parameters) using weighted least squares. The block bootstrap method requires to fix a
size \( \ell \) for the block which typically satisfies \( \ell \to \infty \) and \( \ell/n \to 0 \). For estimating standard errors,
the optimal rate for \( \ell \) is \( n^{1/3} \). See for instance Bühlmann and Künsch (1999). In our simulation
example, we fix \( \ell = 15 \) and the number of bootstrap samples is \( B = 1000 \) and we use 500 replications
to approximate the true probabilities. Smaller or larger blocs did not give much better results here.
Tables 1 and 2 report the coverage probabilities when \( b = cn^{-0.3} \). One can see the accuracy of the
results depend of the bandwidth. The optimal value for the constant seems to be \( c = 1.25 \). Under
the alternative, we compare the power of our test with that of Kim et al. (2015) which consists in
estimating the density of the data normalized by a preliminary estimation of the volatility. We fix
\( \alpha = 10\% \) and the critical values for the competitive test are obtained by simulating their statistics
5000 times under the null hypothesis. Then the level of the second test is exactly fixed. Results
are reported in Table 3. One can see that the results are quite sensitive to the bandwidth. For
our best bandwidth for the coverage probabilities (the constant is \( c = 1.25 \)), we see that the power
of our test is much larger than the optimal power given by the test of Kim et al. (2015). Other
bandwidth parameters did not improve the results. In this conditionally heteroscedastic example,
one can see that working with the original data can be more interesting than first normalizing the
data. However, bandwidth selection seems to be crucial to obtain satisfying results. Note that
bandwidth selection has not been discussed in Kim et al. (2015). Finding results in this direction
is beyond the scope of this paper.

Table 1: Coverage probabilities for the bootstrap approximation (\( n = 300 \))

<table>
<thead>
<tr>
<th>c</th>
<th>1 - ( \alpha ) = 0.90</th>
<th>1 - ( \alpha ) = 0.95</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>97.4</td>
<td>98.6</td>
</tr>
<tr>
<td>0.75</td>
<td>97.2</td>
<td>99.2</td>
</tr>
<tr>
<td>1</td>
<td>95.6</td>
<td>99.2</td>
</tr>
<tr>
<td>1.25</td>
<td>91.2</td>
<td>97.0</td>
</tr>
<tr>
<td>1.5</td>
<td>81.4</td>
<td>91.6</td>
</tr>
</tbody>
</table>
Table 2: Coverage probabilities (in %) for the bootstrap approximation ($n = 600$)

<table>
<thead>
<tr>
<th></th>
<th>c = 0.5</th>
<th>c = 0.75</th>
<th>c = 1</th>
<th>c = 1.25</th>
<th>c = 1.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1 - \alpha = 0.90$</td>
<td>97</td>
<td>95</td>
<td>93.4</td>
<td>90.8</td>
<td>81.8</td>
</tr>
<tr>
<td>$1 - \alpha = 0.95$</td>
<td>99</td>
<td>98</td>
<td>98.2</td>
<td>95.8</td>
<td>91.6</td>
</tr>
</tbody>
</table>

Table 3: Power of the test (in %) for $\delta = 1, n = 600$ and $\alpha = 0.1$

<table>
<thead>
<tr>
<th></th>
<th>c = 0.5</th>
<th>c = 0.75</th>
<th>c = 1</th>
<th>c = 1.25</th>
<th>c = 1.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Test based on $\Delta_n$</td>
<td>6.2</td>
<td>8.6</td>
<td>28.2</td>
<td>70.2</td>
<td>98.4</td>
</tr>
<tr>
<td>Test based on $\Xi_n$</td>
<td>20.4</td>
<td>19</td>
<td>10.4</td>
<td>10</td>
<td>10.6</td>
</tr>
</tbody>
</table>

References


