Violation of transitivity axiom may explain why, in empirical studies, a significant number of subjects violates GARP*

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Abstract

Most papers using experimental data find that a significant number of agents are not utility function maximizers. Using three experimental data sets, we provide empirical evidence that these violations of utility function maximizing behavior are simply generated by a violation of the preference transitivity axiom. Moreover, we find that 97% of the agents' behavior is consistent with maximization of a generalized utility function called variable intervals function (which corresponds to a numerical representation of complete-acyclic preferences).

JEL Codes : C14, D11, D12.

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1 Introduction

The economic theory postulates that individuals have a utility function and choose the best elements with respect to this utility function, under a budget constraint (i.e., what can be chosen is limited by the individuals' revenue). This central assumption is called the *utility maximization hypothesis.* Do agents' behavior in the "real world" consistent with the utility maximization hypothesis ?

A way to answer this question is referred to (in the literature) as the non-parametric method. It relies on an axiomatic approach which states that it is possible to directly test the utility function maximization behavior by testing some axioms.

In order to understand this approach, let us state some definitions from choice functions theory.

Let X be a set of objects, $P(X)$ be the set of subsets of X, and F be a set of non-empty subsets of X. F is called a domain of choice and (X, F) is called a choice space. A domain of choice F is selective if $F \neq P(X) \backslash \varnothing$ and it is abstract otherwise. A (decisive) *choice function* is a function C defined from F to $P(X)$ with the condition that $C(S) \subseteq S$ and $C(S) \neq \varnothing$, $\forall\ S \in F$.

A binary relation Q (over X) rationalizes the choice function C if:

$$\forall\ S \in F,\ C(S) = \{x \in S : xQy,\ \forall\ y \in S\} \tag{1.1}$$

When a binary relation Q rationalizes a choice function, we shall say that the agent (whose choice function is C) respects the preference maximization principle or that he chooses by maximizing his preferences.

A choice function C is *Richter-rational* (Richter 1971) if there exists a binary relation Q (over X) that rationalizes C.

One can remark that such binary relations Q do not need to be complete or transitive.

We can derive the following binary relation R (over X) from a choice function C:

$$\forall\ x, y \in X,\ xRy \quad \text{if} \quad \exists\ S \in F : x \in C(S) \ \ and \ \ y \in S \tag{1.2}$$

Any binary relation Q rationalizing a choice function C includes R. Because of this nice property, Paul Samuelson (see for instance Samuelson 1948) has called *R the revealed preference relation.*

Richter [1971] extends this property by showing that any binary relation Q rationalizing a choice function C is *rationally equivalent* to R:

$$\forall S \in F,\ \{x \in S : xQy,\ \forall y \in S\} = \{x \in S : xRy,\ \forall y \in S\} \tag{1.3}$$

Therefore a choice function is Richter-rational if and only if it is rationalizable by R.

Let us now take $X \subseteq \mathbb{R}^n_+$. If p is a price-vector and m an income, then $B(p,m) = \{x \in X : px \leq m\}$ is called *budget set* is the set of bundles of goods that the agent can afford with a revenue m. Let $F = \{B(p,m)\}_{p,m}$ then the revealed preference relation (see equation 1.2) is written:

$$\forall\ x,\ y \in X,\ x\ R\ y \quad \text{if} \quad \exists\ B(p,m) \in F : x \in C\left(B(p,m)\right) \quad and \quad y \in B(p,m)$$

That is:

$$\forall\ x,\ y \in X,\ x\ R\ y \quad \text{if} \quad \exists\ B(p,m) \in F : x \in C\left(B(p,m)\right) \quad and \quad py \leq m$$

However the economic theory states a condition over binary relations, called *local non-satiation* (see Mas-Collel, Whinston and Green, 1995, chapters 2 and 3) defined by: Let Q be a binary relation defined on $X \subseteq \mathbb{R}^n_+$ and P_Q its asymmetric component; Q is locally non-satiated if for every $x \in X$ and every $\varepsilon > 0$, there exists $y \in X$ such that $\|x - y\| \leq \varepsilon$ and yP_Qx.

This condition says that for any x and for any $\varepsilon > 0$ (a small distance away from x), there exits another bundle of goods y, ε-closed to x with respect to the Euclidian distance, such that y is strictly preferred to x.

The local non-satiation condition implies that over a set $X \subseteq \mathbb{R}^n_+$, the elements that are chosen by the agent necessarily belong to the *budget hyperplan*:

$$\forall\ B(p,m) \in F,\ x \in C\left(B(p,m)\right) \Longrightarrow px = m$$

This property called the *Walras Law*, includes a *desirability property*, which states that the agent is never satiated and his only limit is his revenue.

It is easy to see that when the *Walras law* is satisfied, the revealed preference R writes:

$$\forall\ x,\ y \in X,\ x\ R\ y \quad \text{if} \quad \exists\ B(p,m) \in F : x \in C\left(B(p,m)\right) \quad and \quad py \leq px \tag{1.4}$$

Hence when the *Walras Law* is satisfied, there is no need to specify, in the definition of the revealed preference, the income m of the agent.

Moreover under the *Walras Law*, a binary relation Q (over X) rationalizes the choice function C if (see equation 1.1):

$$\forall\, B(p,m) \in F,\, C\left(B(p,m)\right) = \{x \text{ fulfilling } px = m : xQy,\, \forall\, y \text{ with } py \leq px\}$$

Suppose now that we require such a binary relation Q to be representable by a utility function (order isomorphism) then (under the Walras Law), a utility function rationalizes the choice function C if:

$$\forall\, B(p,m) \in F,\, C\left(B(p,m)\right) = \{x \text{ fulfilling } px = m : u(x) \geq u(y),\, \forall\, y \text{ with } py \leq px\} \tag{1.5}$$

Let us now work with

$$D = \{(x_i, p_i)\}_{i=1}^{N} \tag{1.6}$$

a set of observations of choice bundles of goods $x_i \in \mathbb{R}_+^n$ at corresponding prices $p_i \in \mathbb{R}_+^n$.

Throughout the paper, such a set D defined in (1.6) will be called a *data set.*

Let $X_D = \{x_i : (x_i, p_i) \in D\}$ the support of D (the data set). X_D is the set of bundles that have been chosen by the agent. Nevertheless the choice function of the agent is not necessarily univalent (that is in general, $\forall S \in F, \mid C(S) \mid \geq 1$), hence if a bundle of goods x_i is observed to be chosen at the price vector p_i, this simply means that $x_i \in C\left(B(p_i, p_i x_i)\right)$, not that $x_i = C\left(B(p_i, p_i x_i)\right)$. However $C\left(B(p_i, p_i x_i)\right)$ is not observed by the statistician. As a consequence only the inclusion $\subseteq$ in (1.5), is tested when working over a data set D.

More precisely (see Varian 1982, Chiappori and Rochet 1987), *a (locally non-satiated) utility function u over $\mathbb{R}_+^n$ rationalizes a data set $D = \{(x_i, p_i)\}_{i=1}^{N}$ if:*

$$\forall x_i \in X_D,\ u(x_i) \geq u(y),\, \forall y \in \mathbb{R}_+^n : p_i y \leq p_i x_i \tag{1.7}$$

Finally because for any x_i belonging to X_D, there exists $B(p_i, p_i x_i) \in F$ such that $x_i \in C\left(B(p_i, p_i x_i)\right)$, from (1.4), the revealed preference relation R constructed from the data set D is written:

$$\forall\ x_i \in X_D,\ \forall y \in \mathbb{R}_+^n, \quad x_i\ R\ y \quad \text{if} \quad p_i y \leq p_i x_i \tag{1.8}$$

The interpretation of R is therefore the following: A bundle of goods x_i is *revealed preferred* to y, at the price vector p_i if y could have been chosen by the agent (because y is less than x_i in price) while x_i is chosen.

According to Afriat [1967], it is possible to use this revealed preference approach in order to test whether agents' behavior in the "real world" is consistent with *utility function maximization* (under a budget constraint). More precisely, Varian [1982] shows that his famous Generalized Axiom of Revealed Preference (GARP) is a necessary and sufficient condition for rationalization of a data set D by a (locally non-satiated) utility function.

Given a data set $D = \{(x_i, p_i)\}_{i=1}^{N}$, GARP states that

$$\forall\ x_i,\ x_j \in X_D,\quad x_i\ R^*\ x_j \Rightarrow \mathit{not}(x_j\ RS\ x_i) \tag{1.9}$$

where R^* is the transitive closure of R the *revealed preference* and RS is the so-called *strict revealed preference* defined by:

$$\forall\ x_i \in X_D,\ \forall\ y \in \mathbb{R}_+^n,\qquad x_i\ RS\ y \quad \text{if} \quad p_i y < p_i x_i \tag{1.10}$$

It is important to stress that while R is a relation between[1] X_D and $\mathbb{R}_+^n$, that is a subset of $X_D \times \mathbb{R}_+^n$, the test of GARP (for evident reasons of algorithmic complexity) uses only the restriction of R over X_D.

According to some tests over *experimental data sets*[2] (Sippel 1997, Mattei 2000, Février and Visser 2004 for instance), it seems that agents' behavior, in the "real world", is not consistent with *utility function maximization* (under a budget constraint) as postulated by the economic theory[3].

Sippel [1997] finds in his first experiment with 12 subjects, 5 individuals (41.7%) who violate GARP. He also finds, in his second experiment with 30 subjects, 19 individuals (63.3%) who violate GARP. Mattei [2000] over three experimental data sets obtains the same result: 25, 59 and 51.25 percent of individuals in, respectively, first, second and third Mattei's experimental data sets (consisting of 12, 100 and 320 subjects) violate GARP. Février and Visser [2004] find that 29 % of individuals (35 out of 120) violate GARP.

However it may also be argued that these empirical results simply highlight the fact that preference relations are not transitive in general. Indeed from a mathematical standpoint, a utility function is an order isomorphism. Therefore (over a finite set) representing the preference by a utility function is equivalent to *completeness* and *transitivity* of preference relation. Hence,

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[1] Such a relation is called birelation, see Doignon, Ducamp, Falmagne [1984].

[2] Since the nineties the tests of GARP are made, in general, using data sets constructed by observing (usually in a laboratory) the behavior of agents in choice situations entirely designed and controlled by the experimenter. This is the reason why such data sets are called in the literature *experimental data sets*.

[3] To the best of our knowledge, no paper using experimental data finds zero violation of GARP. Even Andreoni and Miller [2002] who conclude that agents' behavior is consistent with GARP, observe 18 subjects (over a total of 176) who violate GARP.

requiring an agent to maximize a utility function is equivalent to requiring him to maximize a complete and transitive preference relation.

Our argument in this paper is that there is no theoretical or empirical justification for a preference transitivity requirement. This problem is an old one and has its roots in Fechner [1860], Poincaré [1902], or Armstrong [1939] who suggest that an agent's indifference relation (defined as the symmetric part of the corresponding preference relation) is not transitive in general.

The theoreticians of preferences (mostly mathematicians) have therefore constructed several types of non-transitive preference relations with transitive asymmetric component and symmetric component, which are not necessarily transitive. The most famous relation of such a type is the so-called semi-order whose functional representation has been widely studied by Luce [1956], Scott and Suppes [1958], Roberts [1970], Fishburn [1970], Fishburn [1973], Bridges [1983], Chateauneuf [1987], and many others (see Pirlot and Vincke 1997 for a review).

The requirement of transitivity for the asymmetric part of a binary relation is not easy to justify and has been challenged by Kreweras [1961], Burros [1974], Bell [1982], Fishburn [1984], Anand [1987], Fishburn and Lavalle [1987], Loomes et al. [1991], Dombi and Vincze [1994], and many others. Hence, the modeling of preferences by acyclic binary relations, where both symmetric and asymmetric components are not necessarily transitive.

In sum, for most theoreticians of preferences (for a review, see Alekserov, Bouyssou and Monjardet [2007]), the right system of axioms for a preference relation is not *completeness and transitivity* but instead *completeness and acyclicity*. The reason is that on the one hand, having an acyclic preference does not prevent an agent from choosing (this agent can still choose the best elements with respect to his preference) and on the other hand, the acyclicity axiom is compatible with the choice function theory. Indeed, as shown by Jamison and Lau [1973], when the domain of choice is abstract, a choice is Richter-rational (Richter 1971) if and only if it is rationalizable by a complete and acyclic preference relation.

The question we ask in this paper is the following: *do agents' behavior in the "real world" consistent with the maximization of a real-valued function representing a complete-acyclic preference relation?*

In order to answer this question, we show in proposition 1 that a data set D is rationalizable by a (locally non-satiated) complete-acyclic preference if D satisfies an axiom that we call RARP (Richter Axiom of Revealed Preference) which requires over X_D (the support of the data set D) the bilateral asymmetry between the asymmetric part of the aforementioned revealed preference R, denoted P_R, and the transitive closure of P_R, denoted P_R^*.

We test this RARP axiom using three experimental data sets constructed by Mattei [2000]. The first and the second data sets include the consumption behavior of, respectively, 20 and 100 students in 20 different budget situations. The third data set includes 320 subjects who are recruited through an announcement in a Swiss magazine for consumer affairs. We find that the number of individuals who are RARP-consistent represents 100, 97 and 95 percent of individuals in, respectively, the first, second and third experimental data sets. Moreover, in the first data set, 100% of the individuals who violate GARP respect RARP, in the second data set this share is as high as 93%, whereas in the third data set the number of GARP-violating but RARP-consistent individuals reaches 84% of subjects. Thus although over the three data sets more than 30% of individuals are not utility function maximizers, a significant part of them (more than 90% on average) maximize a real-valued function representing a complete-acyclic preference.

The main contributions of this paper consist in (i) providing a sufficient condition for rationalization of the data set by (locally non-satiated) complete-acyclic preferences, (ii) testing this condition over three experimental data sets and (iii) showing that most empirical violations of utility function maximizing behavior are not related to a failure of the maximization behavior but simply generated by a violation of the transitivity axiom.

Our paper is organized as follows. In section two we state basic definitions and we introduce our proposition 1. In the third section, we test RARP over Mattei's experimental data sets. Finally section 4 concludes.

2 A Sufficient Condition for a semi-rationalization of a data set by a variable intervals function

2.1 Binary relations

Let Q be a binary relation over a set X (i.e. Q is a subset of the Cartesian direct product $X \times X$). Like in Alekserov, Bouyssou and Monjardet [2007], we use the notation $(x, y) \in Q$ or xQy to state that the ordered pair (x, y) belongs to Q.

Q can be divided into an *asymmetric component*, denoted P_Q, defined by $\forall x, y \in X, xP_Qy$ if xQy and $not(yQx)$; and a *symmetric component*, denoted I_Q, defined by $\forall x, y \in X, xI_Qy$ if xQy and yQx.

We shall write $Q = P_Q + I_Q$.

Finally $X^2 = P_Q + I_Q + J_Q + P_Q^-$ where J_Q is the "incomparability" relation defined by $\forall x, y \in X, xJ_Qy$ if $not(xQy)$ and $not(yQx)$; and $P_Q^- = \{(x, y) \in X^2 : (y, x) \in P_Q\}$ is the converse of P_Q.

Let us define the following properties of a binary relation Q on the set X:

- $Q^d = \{(x, y) \in X^2 : (y, x) \notin X^2\}$ is the dual relation of Q.
- Q is reflexive if $\forall x \in X, xQx$.
- Q is complete if $J_Q = \emptyset$.
- Q is connected if $\forall x, y \in X, x \neq y, xQy \, or \, yQx$.
- Q is asymmetric if $\forall x, y \in X, xQy \Rightarrow not(yQx)$.
- Q is antisymmetric if $\forall x, y \in X, x \neq y, xQy \Rightarrow not(yQx)$.
- Q is transitive if $\forall x, y, z \in X, xQy$ and $yQz \Rightarrow xQz$.
- Q is negatively transitive if $\forall x, y, z \in X, not(xQy)$ and $not(yQz) \Rightarrow not(xQz)$.
- Q is acyclic if $\forall \, x_1, \ldots, x_k \in X, \; not(x_1 P_Q x_2 P_Q \ldots . P_Q x_k P_Q x_1)$.
- Q is a preorder if Q is complete and transitive.
- Q is an order if Q is reflexive, antisymmetric and transitive.
- Q is a complete order if Q is complete, antisymmetric and transitive.

2.2 Complete-acyclic Preference and Variable intervals functions

The aim of our paper is to check whether the violations of GARP in empirical studies are caused by the fact that agents' preferences are acyclic in general instead of being transitive in general. To do so, we first need to find a numerical representation of complete-acyclic preferences. Fortunately it is well-known since Abbas and Vincke [1993], Agaev and Alekserov [1993], Subiza [1994], Rodriguez-Palmero [1997] and Diaye [1999] that a preference relation defined over a countable[4] set is complete-acyclic if and only if it satisfies the variable intervals model (as defined below).

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[4] In the case of uncountable set, a perfect separability condition has to be added. For instance, suppose that Q is a complete-acyclic relation over an uncountable set X. Then (X, Q) satisfies the Variable Intervals Model iff (X, F) is perfectly separable where $F = P_Q^* \cup J_{P_Q^*}$ with P_Q^* the transitive closure of P_Q and $J_{P_Q^*}$ the incomparability relation w.r.t. P_Q^*.

Definition 1 *Let Q be a binary relation over a set X. (X, Q) satisfies the Variable Intervals Model if there exist two functions (u, s) with $u : X \to \mathbb{R}^n_+$ and $s : X \times X \to \mathbb{R}^n_+$ such that :*

$$\begin{aligned} xP_Qy &\Leftrightarrow u(x) > f(y, x) \\ xI_Qy &\Leftrightarrow u(x) \le f(y, x) \ \text{ and } \ u(y) \le f(x, y) \end{aligned}$$

where $f(x, y) = u(x) + s(x, y)$.

If there is no risk of confusion, we will call such a function f, a *Variable Intervals Function.* If we assume that s is symmetric then we have the below characterization of variable intervals functions by Abbas and Vincke [1993]:

$$\begin{aligned} xP_Qy &\Leftrightarrow u(x) - u(y) > s(x, y) \\ xI_Qy &\Leftrightarrow |u(x) - u(y)| \le s(x, y) \end{aligned}$$

which allows the following interpretation: an agent strictly prefers x to y if the difference of their utilities is greater than a threshold function s which depends on x and y.

2.3 Semi-rationalization of a data set by a variable intervals function

Definition 2 *Let $D = \{(x_i, p_i)\}_{i=1}^N$ be a data set as defined in (1.6). A variable intervals function $f = u + s$ semi-rationalizes D if for any x_i,*

$$\begin{aligned} &(i). && \text{Either: } u(x_i) > u(y) + s(y, x_i) \\ &(ii). && \text{Or: } u(x_i) \le u(y) + s(y, x_i) \ \text{ and } \ u(y) \le u(x_i) + s(x_i, y) \\ &\forall\, y \ \in \ \mathbb{R}^n_+ && \text{such that } \ p_i y \le p_i x_i. \end{aligned}$$

If $f = u$ is a utility function then the definition 2 coincides with the definition of rationalization of a data set by a utility function, used by Varian [1982] (see equation 1.7).

A variable intervals function $f = u + s$ is *non-degenerated* if it fulfills the following condition:

$$P_R \neq \emptyset \Rightarrow \text{not}\, \{u(x_i) \le u(x_j) + s(x_j, x_i)\, \text{and}\, u(x_j) \le u(x_i) + s(x_i, x_j)\, \forall x_i, x_j, x_i \neq x_j\}$$

where P_R is the asymmetric component of R *the revealed preference relation* defined in equation (1.8).

As stressed by Varian [1982, page 946] in the case of rationalization by utility functions, only semi-rationalization by *non-degenerated variable intervals functions* is of interest to us.

Let us now define our Richter Axiom of Revealed Preference.

Definition 3 (RARP) *A data set* $D = \{(x_i, p_i)\}_{i=1}^N$ *satisfies the Richter Axiom of Revealed Preference (RARP) if*

$$\forall\ x_i, x_j \in X_D, i \neq j, \quad x_i\ P_R^*\ x_j \Rightarrow not(x_j\ P_R\ x_i)$$

Where P_R^* *is the transitive closure of* P_R*, the asymmetric component of* R*, the revealed preference relation defined in equation (1.8).*

$GARP$ requires over X_D the bilateral asymmetry between the transitive closure of the revealed preference R and the so-called strict revealed preference denoted RS (see the definition of $GARP$ in equation (1.9)). $RARP$ requires over X_D the bilateral asymmetry between the transitive closure of the asymmetric part P_R of the revealed preference R and the asymmetric part of R. That is, $RARP$ requires over X_D the revealed preference to be acyclic. Note that P_R is by definition asymmetric while this is not the case for RS (under $GARP$, RS is however asymmetric).

$RARP$ is easy to test and its algorithmic complexity is exactly the same as that of $GARP$. The restriction of the revealed preference R over X_D is constructed from the data set D by a direct application of its definition in equation (1.8). finally from this relation, we construct its asymmetric component and the transitive closure of this asymmetric relation (using a least-cost-path algorithm).

Proposition 1 *Let* $D = \{(x_i, p_i)\}_{i=1}^N$ *be a data set. Condition 1 implies Condition 2.*

1. *The data set* D *satisfies* $RARP$.
2. *There exists a (locally non-satiated non-degenerated) variable intervals function which semi-rationalizes the data set* D.

Proof.

Let $X_D = \{x_i : (x_i, p_i) \in D\}$ be the support of D. If the data set $D = \{(x_i, p_i)\}_{i=1}^N$ satisfies $RARP$ then over X_D, R the revealed preference is acyclic. However R was defined in equation (1.8) as follows:

$$\forall\ x_i \in X_D,\ \forall\ y \in \mathbb{R}_+^n, \quad x_i\ R\ y \quad \text{if} \quad p_i y \leq p_i x_i$$

Let $x_i \in X_D$ and $y \in \mathbb{R}_+^n \setminus X_D$ then by definition $x_i\ R\ y$ implies that and $x_i\ P_R\ y$. As a consequence, even if $RARP$ means that "R is acyclic over X_D", $RARP$ implies that R is acyclic over $X_D \cup Y$ where $Y = \{y \in \mathbb{R}_+^n \setminus X_D : \exists\, x_i \in X_D \text{ with } p_i y \leq p_i x_i\}$.

We have $\mathbb{R}^n_+ = X_D \cup Y \cup Z$ where X_D, Y and Z are disjoint sets. X_D is a finite set while Y and Z are uncountable.

Let us extend R which is an acyclic relation over $X_D \cup Y$ into Q a locally non-satiated complete-acyclic relation over $\mathbb{R}^n_+$ by the following way:

1. P_R is included in P_Q and I_R is included in I_Q.
2. $\forall\, x_i \in X_D, \forall\, \varepsilon > 0$, take an element $a \in X_D \cup Z$ such that $\|a - x_i\| \leq \varepsilon$ and state aP_Qx_i.
3. $\forall\, y \in Y, \forall\, \varepsilon > 0$, take an element $b \in X_D \cup Y \cup Z$ such that $\|b - y\| \leq \varepsilon$ and state bP_Qy.
4. $\forall\, z \in Z, \forall\, \varepsilon > 0$, take an element $c \in Z$ such that $\|c - z\| \leq \varepsilon$ and state cP_Qz.
5. $\forall\, x_i, x_j \in X_D$, if $x_iJ_Rx_j$ then $x_iI_Qx_j$.
6. $\forall\, y, y' \in Y$, state yQy' or $y'Qy$ in such a way to preserve acyclicity over Y.
7. $\forall\, x_i \in X_D, \forall\, y \in Y$, if x_iJ_Ry then x_iP_Qy.
8. $\forall\, z, z' \in Z$, state zQz' or $z'Qz$ in such a way to preserve acyclicity over Z.
9. $\forall\, x_i \in X_D, \forall\, z \in Z$, state x_iQz or zQx_i in such a way to preserve acyclicity over $X_D \cup Z$.
10. $\forall\, z \in Z, \forall\, y \in Y$, state zP_Qy.

Steps (2), (3) and (4) can always be performed because the sets Y and Z are uncountable. When steps (2), (3) and (4) are performed, the resulting relation Q is locally non-satiated. Moreover Q is complete-acyclic by construction (see step (1) and steps (5) to (10)).

It is easy to see that there exists a preorder H such that $P_Q \subseteq P_H \subset H \subseteq Q$. For instance, firstly take P_Q^* the transitive closure of P_Q, secondly use the Szpilrajn's theorem in order to extend P_Q^* into a linear order (connected, asymmetric and negatively transitive relation), and thirdly state H as the dual relation of this linear order. Such a H is obviously a complete order.

Let us take a preorder H such that $P_Q \subseteq P_H \subset H \subseteq Q$. Let P_H and I_H be respectively the asymmetric and the symmetric parts of H. There exists a function u defined from $\mathbb{R}^n_+$ to $\mathbb{R}$, such that $\forall\ x,\ y \in \mathbb{R}^n_+$, $xP_Hy \Rightarrow u(x) > u(y)$ and $xI_Hy \Rightarrow u(x) = u(y)$.

From this function u, we construct a function s as follow:

(a.) If xP_Qy then let us take $s(y,x) \in [0,\ \alpha[$.

(b.) If xI_Qy then let us take $s(y,x) \in [\alpha,\ +\infty[$.

where $\alpha = u(x) - u(y)$.

We claim that the resulting function $f(x,y) = u(x) + s(x,y)$ is a variable intervals function representing Q (see definition 1).

Indeed:

(c.) Since $P_Q \subseteq P_H$, then xP_Qy implies that $u(x) > u(y)$. Since $s(y,x) \in [0, u(x) - u(y)[$ then $u(x) > u(y) + s(y,x)$.

(d.) Let $x, y \in \mathbb{R}^n_+$ with $u(x) > u(y) + s(y,x)$. Since Q is complete then either xP_Qy or yP_Qx or xI_Qy. However yP_Qx is contradictory with the statement $u(x) > u(y) + s(y,x)$ because it leads to $u(y) > u(x)$. xI_Qy is also contradictory with the statement $u(x) > u(y) + s(y,x)$ (which can be rewritten $s(y,x) < u(x) - u(y)$) because it leads to $s(y,x) \geq u(x) - u(x)$ (see the above point (b)).

(e.) Let $x, y \in \mathbb{R}^n_+$ with xI_Qy, then either $u(x) < u(y)$, $u(x) > u(y)$ or $u(x) = u(y)$. If $u(x) < u(y)$ then $u(x) < u(y) + s(y,x)$ and $u(y) \leq u(x) + s(x,y)$. If $u(x) > u(y)$ then $u(y) < u(x) + s(x,y)$ and $u(x) \leq u(y) + s(y,x)$. If $u(x) = u(y)$ then $u(y) \leq u(x) + s(x,y)$ and $u(x) \leq u(y) + s(y,x)$.

(f.) Let $x, y \in \mathbb{R}^n_+$ with $u(x) \leq u(y) + s(y,x)$ and $u(y) \leq u(x) + s(x,y)$. Either xP_Qy, yP_Qx or xI_Qy. However xP_Qy and yP_Qx are impossible because they lead respectively to $u(x) > u(y) + s(y,x)$ and $u(y) > u(x) + s(x,y)$ (see point (c) above).

Points (c) and (d) prove that $xP_Qy \Leftrightarrow u(x) > f(y,x)$ while points (e) and (f) prove that $xI_Qy \Leftrightarrow u(x) \leq f(y,x)\, and\, u(y) \leq f(x,y)$.

Since Q is an extension over $\mathbb{R}^n_+$ of R the revealed preference relation (as defined in (1.8)), then it must be the case that f semi-rationalizes the data set D (see definition 2).

Such a function f is locally non-satiated (and non-degenerated) because we have constructed Q in such a way. ■

3 Tests of RARP over Mattei's experimental data sets

The purpose of our tests is to distinguish, among the RARP-rational individuals those whose behavior is consistent with utility function maximization (that is, who are GARP-consistent) or with variable intervals function maximization (that is, who are RARP-consistent).

3.1 The data sets

We use three experimental data sets constructed by Mattei [2000]. These data sets include respectively 20, 100, and 320 individuals who have to choose among 8 goods in 20 different budget situations (i.e., 20 different budget sets $B(p, m)$, see Section 2.3).

In the first and second data sets, the goods are: *milk chocolate, salted peanuts, biscuits, text maker, ball-point pen, plastic folder, writing pad, and post it.* In the third data set, the goods are: *milk chocolate, biscuit, orange juice, iced tea, post it, audio cassette c90, ball point pen, and battery (R6, 1.5V).*

Hence each experimental data set represents in fact a collection of data sets $D = \{(x_i, p_i)\}_{i=1}^{N}$ as defined in (1.6) with:
$N = 20$ different budget situations,
$x_i = (x_{i1}, \cdots, x_{in})$ with $n = 8$ goods.

For instance the first, second and third experimental data sets are respectively a collection of 20, 100 and 320 data sets (one per individual) as defined in (1.6).

In the first experiment, the individuals are students from a microeconomics class and the amount of money to spend per student in each budget situation varies between 30 and 40 Swiss Franc. In the second experiment, the individuals are business students and the amount of money to spend per student in each budget situation varies between 42 and 56 Swiss Franc. In order to incite the individuals to participate in the experiments and to behave as real consumers, they were informed that they could receive one of these bundles of goods. The experiments are performed in a computer center in which all the information is available on the computers screens, especially the price of the good. The individuals have simply to fill in the quantities they want to buy in each budget situation.

In the third experiment, the individuals are recruited through an announcement in a Swiss magazine for consumer affairs. A questionnaire where the participants have to write down their choices was sent to the individuals who answered as being interested in the experiment.

3.2 Robustness of the tests

Before presenting the results, we need to point out that since the tests are non-parametric, it is important to pay particular attention to their robustness.

Firstly, we need to take into account the "trivial respect" and "trivial violation" of the axioms.

A trivial respect of the axioms arises when given (x_i, x_j) , the total expenditure of x_i evaluated at a reference price p_0 is sharply "greater" than the one of x_j evaluated at the same reference price. Indeed in such a case, we have $p_i x_i > p_i x_j$ and $p_j x_i > p_j x_j$. In this paper, because we use experimental data, we avoid such kind of problem.

(x_i, x_j) trivially violates the axioms if the total expenditures of x_i and x_j evaluated at the available price p_i are such that that is their difference is smaller than δ an arbitrarily small positive real value. In such a case, the bundles x_i and x_j could be considered, by the agent, as identical and thus we may have $p_i x_i \geq p_i x_j$ and $p_j x_j > p_j x_i$. In order to take into account possible trivial violations, Afriat [1967] suggests using the binary relations[5] R_e and RS_e :

$$\forall\ x_i,\ x_j, \quad x_i\ R_e\ x_j \Leftrightarrow p_i x_j \leq e \times p_i x_i \tag{3.1}$$

$$\forall\ x_i,\ x_j, \quad x_i\ RS_e\ x_j \Leftrightarrow p_i x_j < e \times p_i x_i \tag{3.2}$$

where $e \in [0,1]$ is known as the *Afriat Efficiency Index* (Varian 1990).

Secondly, we need to evaluate the power of the tests. From a statistical viewpoint the power of a test between two assumptions is the probability of rejecting the null hypothesis while the alternative is true. In our case, the former is that the consumer behavior satisfies the axiom we test, and the latter that it does not. Unfortunately, since non-parametric tests of GARP and RARP are non probabilistic, their power is unknown.

A consensus in the literature however exists concerning the test of GARP: it consists in computing an approximate power of the nonparametric test. Among various methods for approximation of power on non-parametric tests (see Andreoni and Harbaugh 2006 for a detailed review), the one by Bronars [1987] is most extensively used. Following Becker [1962], Bronars computes the approximate power of the test by taking the assumption on the randomness of individuals' choices as alternative hypothesis.

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[5] Let us point out that R_e is included in R, however its asymmetric part is not necessarily included in the asymmetric part of R.

The rationale of the Bronars' method is that the randomly generated data expresses in some sense the behavior of "erratic" individuals who choose randomly. As a consequence, the behavior of individuals in the initial data should be different from the behavior of the "erratic" individuals. In particular, the number of violations of GARP over the randomly generated data should be higher than the number of violations of GARP over the initial data.

Actually Bronars [1987, pages 695-696] proposes three methods. Let $D = \{(x_i, p_i)\}_{i=1}^N$ with $x_i = (x_{i1}, \cdots, x_{in})$ and $p_i = (p_{i1}, \cdots, p_{in})$. The first method proposed by Bronars is to generate a number K of random consumption uniformly distributed in the budget set. These set are $\widetilde{D}_k = \left\{\left(\widetilde{x}_i^k, p_i\right)\right\}_{i=1}^N$, $k = 1 \cdots K$, where $\widetilde{x}_i^k = \left(\frac{s_1 \times p_i x_i}{p_{i1}}, \cdots, \frac{s_n \times p_i x_i}{p_{in}}\right)$ is a random consumption, p_i is the observed price and the $s_1, \cdots, s_n$ are drawn from a $[0, 1]$ uniform law in such a way that $\sum_{j=1}^n s_j = 1$.

Each s_j is called budget share of good j, that is the amount of the budget devoted to the consumption of good j.

One can remark that in the first method of Bronars, the budget shares are completely arbitrary. The second method imposes the expected budget shares to be equal to $1/n$. More precisely, $s_j = \frac{\widetilde{z}_{ij}^k}{\sum_{j=1}^n \widetilde{z}_{ij}^k}$ where the $\widetilde{z}_{i1}^k, \cdots, \widetilde{z}_{in}^k$ are drawn from a uniform law.

In his third method, Bronars suggests the random budget shares to mimic the observed budget shares. That is to state s_j as:

$$s_j = \gamma_j \frac{\widetilde{z}_{ij}^k}{\sum_{j=1}^n \gamma_j \widetilde{z}_{ij}^k}$$

where γ_j is the mean budget share of good j in the data set $D = \{(x_i, p_i)\}_{i=1}^N$.

The Bronars (method 1, 2 or 3) power index is defined as the average percentage of GARP violating individuals over the K fictive random consumption data exhausting the budget set.

Mattei [2000] presented the results concerning the Bronars Method 2 power index for the test of GARP. Since we use in this paper the experimental data sets of Mattei [2000], we present (in order to be able to compare) the results concerning the Bronars Method 2 power index. We call this index, the *absolute Bronars power index.* We use also the Bronars' method 2 in order to calculate the power of the RARP test.

Unfortunately, as remarked by Andreoni and Harbaugh [2006, page 6], *"An advantage of Bronars' approach is that it is both natural and simple. A disadvantage is that the alternative hypothesis is perhaps too naive."*.

Moreover the Bronars methods 1, 2 and 3 can provide very different results. For instance, in their 2002 paper in Econometrica, Andreoni and Miller reports a Bronars Method 1 Power Index of 0.78 for the tests of GARP. However according to Andreoni and Harbaugh [2006], over the same data set, the Bronars Method 2 power index and the Bronars Method 3 power index are respectively of 0.63 and 0.48 for the GARP test. This shows the sensitivity of the power test to the alternative hypothesis.

It is beyond the scope of this paper to discuss the right method to calculate the power of non-parametric tests. We want simply to point out that RARP is a very weak axiom, so both over the initial data and the randomly generated data, the number of violations of RARP will mechanically be weak. This is the reason why we define also a *relative Bronars power index* as the relative increase[6] of the number of GARP or RARP violating individuals when testing the axioms over the data set D and over the random consumption data sets $\widetilde{D}_k$. Using this relative power index, one can see whether the behavior of the individuals from the initial data sets is "significantly" different from the behavior of the individuals from the randomly generated data sets.

3.3 Results

In order to take into account possible errors of optimization and/or measurement, we provide, in tables 1 to 3, the results for different values of the Afriat efficiency index.

Over the first data set (Table 1):

- The behavior of 75% of the subjects is consistent with utility function maximization: their behavior satisfies GARP.
- The behavior of 100% of the subjects is consistent with variable intervals function maximization: their behavior satisfies RARP.

Table 1

Number of GARP, and RARP-irrationals subjects over the first data set (20 subjects).

Afriat Index	GARP	RARP
1.00	5	0
0.99	2	0
0.98	1	0
0.97	1	0

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[6] Formally, the relative Bronars power index writes $\frac{nv-nvr}{nv}$ where nv is the number of violating individuals in the data set D and nvr is the average number of individuals over the random consumption data sets $\widetilde{D}_k, \quad k = 1 \cdots K$.

Over the second data set (Table 2):

- The behavior of 56% of the subjects is consistent with utility function maximization: their behavior satisfies GARP.
- The behavior of 97% of the subjects is consistent with variable intervals function maximization: their behavior satisfies RARP.

Table 2

Number of GARP, and RARP-irrationals subjects over the second experimental data set (100 subjects).

Afriat Index	GARP	RARP
1.00	44	3
0.99	30	2
0.98	16	1
0.97	11	1

Over the third data set (table 3):

- The behavior of 68.43% of the subjects is consistent with utility function maximization: their behavior satisfies GARP.
- The behavior of 95% of the subjects is consistent with variable intervals function maximization: their behavior satisfies RARP.

Table 3

Number of GARP, and RARP-irrationals subjects over the third experimental data set (320 subjects).

Afriat Index	GARP	RARP
1.00	101	16
0.99	66	10
0.98	50	5
0.97	35	6

3.4 Power analysis

We provide here the results[7] concerning the *absolute Bronars power index* and the *relative Bronars power index.*

Let us remind (see section 3.2) that the *absolute Bronars power index* is defined as the average percentage of GARP or RARP violating individuals over K (we take here $K = 10,000$) fictive random consumption data

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[7] We have also computed another power index called *Afriat Confidence Index* (see Andreoni and Harbaugh 2006). The results are available upon request.

exhausting the budget set; and that the *Relative Bronars Power Index* is defined as the relative increase of the number of (RARP, GARP) violating individuals when testing the axioms over the data set and over the random consumption data sets.

The absolute Bronars Power Index of RARP is, respectively, of 51.3%, 56.17% and 55.57% for the first data set, the second data set and the third data set.

Moreover, the Relative Bronars Power Index of RARP is on average 13.5. The meaning is that from the data sets to the random consumption data sets, the number of RARP violating individuals increases on average by 1350%.

Table 4

Absolute Bronars Power Index and Relative Bronars Power Index (result in brackets) over the first experimental data set

Afriat Index	GARP	RARP
1.00	0.9745 (2.898)	0.513 (.)
0.99	0.9465 (8.465)	0.412 (.)
0.98	0.8765 (16.53)	0.3295 (.)
0.97	0.748 (13.96)	0.235 (.)

Trial: K=10,000

Table 5

Absolute Bronars Power Index and Relative Bronars Power Index (result in brackets) over the second experimental data set

Afriat Index	GARP	RARP
1.00	0.9896 (1.24)	0.5617 (17.72)
0.99	0.9606 (2.20)	0.4477 (21.38)
0.98	0.8836 (4.52)	0.3381 (32.81)
0.97	0.7619 (5.92)	0.2263 (21.63)

Trial: K=10,000

Table 6

Absolute Bronars Power Index and Relative Bronars Power Index (result in brackets) over the third experimental data set

Afriat Index	GARP	RARP
1.00	0.98888 (2.13)	0.55572 (10.11)
0.99	0.95716 (3.64)	0.44106 (13.11)
0.98	0.88119 (4.63)	0.32809 (19.99)
0.97	0.75109 (5.86)	0.22184 (10.83)

Trial: K=10,000

4 Discussion and Conclusion

Our results (see tables 1, 2 and 3 in Section 3.2) reveal that over three data sets, the number of individuals who are *RARP* consistent represents, respectively, 100, 93 and 84 percent of individuals whose behavior is not consistent with utility function maximization (that is, who are not *GARP* consistent). Thus, although the behavior of more than 30 percent of individuals is not consistent with utility function maximization, the behavior of a great part of them (more than ninety percent on average) is consistent with variable intervals function maximization.

In other words, we show that most empirical violations of *GARP* reported in the economic literature, are simply generated by violation of transitivity.

This empirical finding is important because it suggests a rational individual may be defined as an individual who is a variable intervals function maximizer. And therefore this may be a call for a consumer theory reconstruction based on complete-acyclic preferences.

The main limitation of our work is the power of the RARP tests. Indeed using the Bronars' method 2, we find on average a power of 54%, that is 54% of the individuals in the random consumption data sets, violate RARP. However this figure has to be compared to the percentage of RARP violating individuals over the data sets (3% on average).

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